ELECTRICAL RESISTANCE OF THE LOW DIMENSIONAL CRITICAL BRANCHING RANDOM WALK

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Abstract. We show that the electrical resistance between the origin and generation n of the incipient infinite oriented branching random walk in dimensions $d < 6$ is $O(n^{1-\alpha})$ for some universal constant $\alpha > 0$. This answers a question of Barlow, Járai, Kumagai and Slade [2].

1. Introduction

We study the electrical resistance of the trace of oriented critical branching random walk (BRW) in low dimensions. This trace is obtained by drawing a critical Galton-Watson tree $\mathcal{T}$ conditioned to survive forever and randomly mapping it into $\mathbb{Z}^d \times \mathbb{Z}^+$ in the following manner: we initialize by mapping the root of $\mathcal{T}$ to $(o, 0)$ and recursively, if $V \in \mathcal{T}$ was mapped to (x, n) and $U \in \mathcal{T}$ is a child of V, then we map U to $(y, n+1)$ where y is chosen according to a symmetric random walk distribution (we assume that this distribution has an exponential moment). Denote by $\Phi : \mathcal{T} \to \mathbb{Z}^d \times \mathbb{Z}^+$ this random mapping. The trace we consider in this paper is the graph induced by set of edges $\{\Phi(V), \Phi(U)\}$ for every edge $\{U, V\}$ of $\mathcal{T}$.

It follows from the work of Barlow, Járai, Kumagai and Slade [2, Example 1.8(iii)] (who studied the much more difficult model of critical oriented percolation (OP)) that when $d > 6$, the electrical resistance between the root and generation n in the BRW is linear in probability. This enabled them to calculate various exponents describing the behavior of the simple random walk on the trace. In particular, they show that the mean hitting time of graph distance n is $\Theta(n^3)$, that the spectral dimension equals $4/3$ and more, see [2].

They asked [2, Section 1.4, Example 1.8 (iii)] whether the resistance of the critical BRW is still linear in n in dimensions $4 < d \leq 6$, that is, in any dimension above the critical dimension 4 of OP [6, 7, 8, 9]. Here we answer their question by showing that the resistance is $O(n^{1-\alpha})$ when $d \leq 5$.

Theorem 1.1. *Let $R(n)$ denote the expected effective resistance between the origin and generation n of a branching random walk in dimension $d < 6$ with progeny distribution that has mean 1, positive variance and finite third moment, conditioned to survive forever. Assume that the random walk steps are symmetric, non-degenerate and have exponential tails. There exists a* *universal constant $\alpha > 0$ such that*

$$R(n) = O(n^{1-\alpha})\,.$$

Unlike our firm understanding of anomalous diffusion in high dimensions [2, 11, 13], random fractals in low dimensions are not (stochastically) finitely ramified. That is, we do not see pivotal edges at every scale. This makes their analysis more challenging, even in the case of the critical BRW which is one of the simplest models of statistical physics. Our argument heavily relies on the built-in independence and self-similarity of the model to obtain recursive inequalities for the resistance. We first show that intersections within the trace occur at every scale (see Figure 1 and Theorem 2.1); these intersections exist only when $d < 6$. Secondly, we show that the branches leading to each intersection are themselves distributed as BRW, allowing us to bound the electrical circuit using the parallel law and to form recursive estimates (Theorem 2.2). There are additional technical difficulties to overcome. For instance, when intersections do not occur, the resistance is stochastically larger than it is unconditionally and one needs to get adequate bounds on it. Calculating the precise polynomial exponent which determines the growth of $R(n)$ when $d < 6$ remains a challenging open problem.

As mentioned before, it is believed that OP in $d = 5$ behaves similarly to BRW hence we expect an analogue of Theorem 1.1 to hold. Presumably, the general setup (illustrated in Figure 1) and proving existence of intersections (Theorem 2.1) can be done for OP (based on results of [6, 7]). However, due to the lack of distributional self-similarity in OP it seems difficult to obtain recursive bounds (that is, an analogue of Theorem 2.2). Furthermore, we do not know whether the exponent determining the growth of the resistance in OP in $d = 5$ should be the same as the one for BRW (assuming they both exist).

It is easy to see (and stated in [2]) that the volume up to generation n of the BRW trace is of order $\Theta(n^2)$ in probability. Hence, Theorem 1.1 together with the commute time identity (1.1) shows that the mean exit time of the simple random walk on the BRW trace from the ball of radius n in graph distance is at most $O(n^{3-\alpha})$, i.e., much faster than the $\Theta(n^3)$ in dimensions $d > 6$, see [2]. In fact, if one calculated the exponent determining the growth of the resistance, then many other random walk exponents (such as the spectral dimension, walk dimension etc.) could be determined, see [14]. In particular, if the resistance exponent exists, it follows from our results that the spectral dimension is strictly larger than 4/3.

Remark 1. We emphasize that the exponent $\alpha > 0$ of Theorem 1.1 is universal in the sense that it does not depend on the progeny or random walk distributions.

Remark 2. By projecting the trace to $\mathbb{Z}^d$ we get a similar result for the usual (non-oriented) branching random walk: the effective resistance between the origin and the particles of generation n is $O(n^{1-\alpha})$ when $d \le 5$.

This is because the projection only decreases the effective resistance. By a similar argument, projecting $\mathbb{Z}^5$ into $\mathbb{Z}^d$ with $d < 5$, we learn that it suffices to prove Theorem 1.1 for $d = 5$.

1.1. **Incipient infinite branching process.** Let $\{p(k)\}_{k \geq 0}$ be a progeny distribution of a Galton-Watson branching process. Our assumptions on $\{p(k)\}$ are the following.

(i) Criticality: $\sum_k kp(k) = 1$.
(ii) Finite variance: $\sum_k k(k-1)p(k) = \sigma^2 \in (0, \infty)$.
(iii) Bounded third moment: $\sum_k k^3 p(k) \leq C_3 < \infty$

The variance σ^2 enters our arguments as a time scale parameter. In particular, we use the well-known result of Kolmogorov [12, 1] that the survival probability to time n is asymptotic to $2(\sigma^2 n)^{-1}$. Our assumption on the third moment is mainly for convenience: it gives easy control over the rate of convergence in the asymptotics of the survival probability (see the estimates in Section 3.1). We have not attempted to optimize the moment condition on p.

It is classical that under condition (i) (and that $p(1) < 1$) the branching process dies out with probability 1. To construct the incipient infinite branching process (IIBP), we simply condition on survival up to level n, and take the weak limit of the measures obtained as $n \to \infty$. However, it will be convenient for us to use an equivalent construction of the IIBP (see [11, 16]).

Consider an infinite path $(V_0, V_1, \ldots)$ and attach to each vertex V_i a critical branching process with progeny distribution $\tilde{p}$ in the first generation and p afterwards, where $\tilde{p}$ is the law of $X - 1$ where X has the size biased law of p, in other words,

$$\tilde{p}(k) = (k+1)p(k+1) .$$

1.2. **Incipient infinite branching random walk.** Let $\mathbf{p}^1(x, y)$ denote the 1-step transition probability of a random walk on $\mathbb{Z}^d$. We assume the following:

(i) Exponential moment: $\sum_{x \in \mathbb{Z}^d} e^{b|x|} \mathbf{p}^1(o, x) < \infty$ for some $b > 0$.
(ii) Non-degeneracy: $\{x \in \mathbb{Z}^d : \mathbf{p}^1(o, x) > 0\}$ generates $\mathbb{Z}^d$ as a group.
(iii) Symmetry: $\mathbf{p}^1(x, y) = \mathbf{p}^1(y, x)$.

We remark that we did not try to obtain the optimal condition on $\mathbf{p}^1$. In fact, conditions (ii) and (iii) are not essential for our proof, and (i) can plausibly be replaced with a weaker condition, however, we opted to make the calculations smoother.

Given a rooted tree T we define a random mapping $\Phi : T \to \mathbb{Z}^d \times \mathbb{Z}_+$ which we will call henceforth a "random walk" mapping. Firstly, Φ maps the root of T to $(o, 0)$ and recursively, given a vertex V of T at height h and its mapping $\Phi(V) = (x, h)$ we map each upward neighbor U of V, independently, by drawing a random neighbor y of x, according to $\mathbf{p}^1(x, \cdot)$, and putting $\Phi(U) = (y, h+1)$. The *incipient infinite branching random walk* (IIBRW) is obtained by taking T to be the IIBP.

For any tree T we consider $\Phi(T)$ as a graph on the vertex set $\mathbb{Z}^d \times \mathbb{Z}_+$ and we add the edge $\{\Phi(U), \Phi(W)\}$ for any tree edge $\{U, W\}$ (there may be parallel edges). The *trace* of the IIBRW is simply $\Phi(T)$ where T is the IIBP.

1.3. **Electrical resistance.** We provide a brief background on the electric effective resistance of a network, for further information see [15]. Let $G = (V, E)$ be a finite connected graph with two marked vertices a and z (we assume here that all edge weights are 1). Recall that a unit flow θ from a to z is an antisymmetric function on the edges E such that if we put $\theta_v = \sum_{x:(x,v)\in E} \theta(vx)$, then $\theta_a = 1, \theta_z = -1$ and for all other vertices $v \neq a, z$ we have $\theta_v = 0$.

The *effective resistance* between a and z, denoted $R_{\text{eff}}(a \leftrightarrow z)$, is the minimum energy $\mathcal{E}(\theta)$ over all unit flows θ from a to z, where $\mathcal{E}(\theta) = \sum_{e\in E} \theta(e)^2$. The connection between this quantity and the simple random walk $\{S_n\}_{n\geq 0}$ on G is evident via the identity (see [15]),

$$R_{\text{eff}}(a \leftrightarrow z) = \frac{1}{\deg(a)\mathbf{P}_a(\tau_z < \tau_a^+)} ,$$

where $\deg(a)$ is the vertex degree of a, τ_z is the first visit time to z, τ_a^+ is the first positive visit time to a and $\mathbf{P}_a$ is the simple random walk probability measure conditioned on $X_0 = a$. Another useful connection is the *commute time identity* asserting that

$$\mathbf{E}_a\tau_z + \mathbf{E}_z\tau_a = 2|E|R_{\text{eff}}(a \leftrightarrow z) . \tag{1.1}$$

We will frequently use the easy fact that the resistance satisfies the triangle inequality, that is, for any three vertices x, y, z we have

$$R_{\text{eff}}(x \leftrightarrow z) \leq R_{\text{eff}}(x \leftrightarrow y) + R_{\text{eff}}(y \leftrightarrow z) . \tag{1.2}$$

Lastly, we will use the *parallel law* for effective resistance stating that if $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ are two connected graphs on the same vertex set and $a, z \in V$, then the effective resistance $R_{1\cup 2}$ between a and z in $(V, E_1 \cup E_2)$ (where we allow multiple edges in this union) satisfies

$$\frac{1}{R_{1\cup 2}} \geq \frac{1}{R_1} + \frac{1}{R_2} , \tag{1.3}$$

where R_1, R_2 are the effective resistances between a and z in G_1, G_2, respectively.

1.4. **Finite approximations.** We use the following finite approximations to the IIBRW in order to establish recursions.

Definition. Suppose $n \geq 1$ and $m \geq 2n$. Let $\mathcal{T}_{n,m}$ denote the following random tree:

(i) A path of length n (the *backbone*): $(V_0, \dots, V_n)$ with a marked root $\rho = V_0$.

(ii) For each $0 \le i \le n-1$ attach to V_i a critical branching process with progeny distribution $\tilde{p}$ in the first generation and p afterwards, conditioned to die out before generation $m - i$ (i.e. none of the vertices of the attached trees reach distance m from ρ).

The following is an important quantity in the proof. For $x \in \mathbb{Z}^d$ define

$$\gamma(n, x) = \sup_{m \ge 2n} \mathbf{E}_{\mathcal{T}_{n,m}} \left[R_{\text{eff}}\big((o,0) \leftrightarrow \Phi(V_n)\big) \mid \Phi(V_n) = (x, n) \right], \tag{1.4}$$

where we consider the resistance in the graph $\Phi(\mathcal{T}_{n,m})$.

It will be convenient to introduce the following norm on $\mathbb{R}^d$ adapted to the "typical size" of the random walk displacements. We do this in order to conveniently obtain a universal estimate on α of Theorem 1.1, but the reader may just assume that $\mathbf{p}^1$ is the transition matrix of the nearest-neighbor simple random walk and that the norm below is the Euclidean norm.

Let $Q_{ij} = \sum_{x \in \mathbb{Z}^d} x_i x_j \mathbf{p}^1(o, x)$ be the covariance matrix of the step distribution, and let Q^{-1} denote the inverse of Q. We define

$$\|x\| := \sqrt{\frac{1}{d} \sum_{i,j=1}^{d} x_i Q^{-1}_{ij} x_j}\,. \tag{1.5}$$

The main effort in this paper is the following theorem.

Theorem 1.2. *Assume $d \le 5$. There exists a universal constant $\alpha \in (0, 1/2)$ and also $A = A(\sigma^2, C_3, \mathbf{p}^1) < \infty$ such that for all $n \ge 1$*

$$\gamma(n, x) \le A n^{1-\alpha} \Big(\frac{\|x\|^2}{n} \vee 1 \Big)^{\alpha}.$$

Remark. Note that we cannot expect $\gamma(n, x)$ to be $O(n^{1-\alpha})$ for all x. Indeed, when $\|x\| = \Theta(n)$, then conditioned on $\Phi(V_n) = (x, n)$ the projection of the path $\Phi(V_0), \dots, \Phi(V_n)$ onto $\mathbb{Z}^d$ has positive speed and since this conditioning does not affect the mapping of the trees hanging on V_i, there will be few intersections and we expect the resistance then to be linear in n. Theorem 1.2 will be proved by induction, hence it has to contain an estimate valid for all $x \in \mathbb{Z}^d$.

Proof of Theorem 1.1 assuming Theorem 1.2. Recall that in the construction of the IIBRW we attach to the backbone unconditional critical trees, whereas in the definition of $\mathcal{T}_{n,m}$ we attach critical trees conditioned not to reach a certain level. However, when n is fixed and $m \to \infty$ the distribution of these critical trees tends to the distribution of an unconditional critical tree. Hence,

$$R(n) \le \lim_{m \to \infty} \mathbf{E}_{\mathcal{T}_{n,m}} \left[R_{\text{eff}}\big((o,0) \leftrightarrow \Phi(V_n)\big) \right],$$

where we have bounded the resistance to generation n by the resistance to a single vertex $\Phi(V_n)$. Therefore,

$$R(n) \le \sup_{m \ge 2n} \mathbf{E}_{\mathcal{T}_{n,m}} \big[R_{\mathrm{eff}}\big((o,0) \leftrightarrow \Phi(V_n)\big)\big] = \sum_{x \in \mathbb{Z}^d} \mathbf{p}^n(o,x)\gamma(n,x)\,.$$

By Theorem 1.2 we have

$$R(n) \le A n^{1-\alpha} \sum_{x : \|x\| \le \sqrt{n}} \mathbf{p}^n(o,x) + A n^{1-\alpha} \sum_{x : \|x\| \ge \sqrt{n}} \mathbf{p}^n(o,x) \Big(\frac{\|x\|^2}{n} \Big)^{\alpha}.$$

The first sum is bounded by $An^{1-\alpha}$. For the second sum we bound by

$$\sum_x \mathbf{p}^n(o,x) \Big(\frac{\|x\|^2}{n} \Big) = 1\,,$$

(see Section 1.5) concluding the proof. $\square$

1.5. **Some random walk estimates.** We provide here some standard random walk estimates that will be useful throughout the proof. We denote by $\{S(n)\}_{n \ge 0}$ a random walk with step distribution $\mathbf{p}^1$ and $S(0) = o$. Let $(S^1(n), \dots, S^d(n))$ denote the coordinates of $S(n)$ in a coordinate system that diagonalizes Q^{-1} (lower indices will be used for the Euclidean coordinates). Due to independent and mean zero increments and the definition of the norm, we have

$$\mathbf{E}\Big[\|S(n)\|^2 \Big] = n \mathbf{E}\Big[\|S(1)\|^2 \Big] = \frac{n}{d} \sum_{i,j=1}^{d} \mathbf{E}\Big[S(1)_i Q^{-1}_{ij} S(1)_j \Big] = n\,.$$

Applying Chebyshev's inequality, we get

$$\sum_{x \in \mathbb{Z}^d : \|x\| \le \sqrt{2}} \mathbf{p}^1(o,x) = 1 - \mathbf{P}(\|S(1)\|^2 > 2) \ge 1/2\,. \tag{1.6}$$

The central limit theorem [3, Theorem 2.9.6] implies that for any $0 < L < \infty$ and any $v \in \mathbb{R}^d$, we have

$$\lim_{n \to \infty} \mathbf{P}(\|S(n) - \sqrt{n}v\| \le L\sqrt{n}) = C(v,d,L) \in (0,1)\,, \tag{1.7}$$

with the constant $C(v,d,L)$ independent of $\mathbf{p}^1$.

The following proposition summarizes some estimates we will need on the random walk S conditioned on the event $\{S(n) = x\}$.

Proposition 1.3. *There exists $k_1 = k_1(\mathbf{p}^1)$, $C > 0$ and $\delta_1 = \delta_1(d) > 0$ such that the following hold.*
(i) Whenever $k_1 \le k \le n$, $\|x\| \le 4n/\sqrt{k}$, we have

$$\mathbf{E}\Big[\|S(k)\|^2 \,\Big|\, S(n) = x \Big] \le Ck\,. \tag{1.8}$$

(ii) Whenever $k_1 \le k \le \delta_1 n$, $\|x\| \le 4n/\sqrt{k}$, we have

$$\mathbf{E}\Big[\|S(k)\|^2 \,\Big|\, S(n) = x,\ \|S(k)\| > \sqrt{k} \Big] \le Ck\,. \tag{1.9}$$

(iii) Whenever $k_1 \le k \le \delta_1 n$, $k_1 \le k' \le n-k$ *and* $\|x\| \le \min\{4n/\sqrt{k}, 4n/\sqrt{k'}\}$, *we have*

$$\mathbf{E}\Big[\|S(k+k') - S(k)\|^2 \,\Big|\, S(n) = x,\ \|S(k)\| > \sqrt{k}\Big] \le Ck' . \tag{1.10}$$

For the proof of this proposition, we will use the exponential moment assumption from Section 1.2. Let $b_1 > 0$ be such that when $\|\beta\| < b_1$ we have

$$Z_\beta := \sum_{y \in \mathbb{Z}^d} e^{\beta \cdot y} \mathbf{p}^1(o, y) < \infty ,$$

where $\cdot$ in the exponent denotes inner product with respect to the quadratic form $\sum_{i,j} x_i Q_{ij}^{-1} y_j$. Define the exponentially tilted step distribution

$$\mathbf{p}^1_\beta(o, y) = \frac{1}{Z_\beta} e^{\beta \cdot y} \mathbf{p}^1(o, y)$$

Let $X(1), \dots, X(n)$ be i.i.d. distributed according to $\mathbf{p}^1$, so that $S(n) = X(1) + \cdots + X(n)$, and let $X_\beta(1), \dots, X_\beta(n)$ be i.i.d. distributed according to $\mathbf{p}^1_\beta$. Let $S_\beta(n) = X_\beta(1) + \cdots + X_\beta(n)$ and let

$$m_\beta := \mathbf{E}[X_\beta(1)] = \frac{1}{Z_\beta} \sum_{y \in \mathbb{Z}^d} y e^{\beta \cdot y} \mathbf{p}^1(o, y) = \nabla \log Z_\beta .$$

Since the Jacobian of $\beta \mapsto m_\beta$ at $\beta = 0$ is non-singular, for $v \in \mathbb{R}^d$ sufficiently close to 0 there exists a unique β such that $m_\beta = v$. We write Q_β for the covariance matrix of $X_\beta(1)$, $D_\beta = \det(Q_\beta)^{1/2d}$, and $\|\cdot\|_\beta$ for the norm arising from Q_β^{-1}. Note that D_β and $\|\cdot\|_\beta$ depend continuously on β in a neighbourhood of 0. In particular, for β in a neighbourhood of 0 we have

$$\|v\|_\beta \le 2\|v\| . \tag{1.11}$$

Since $\mathbf{E}[\|X(1)\|^2] = 1$, for β sufficiently close to 0, we have

$$\Sigma_\beta^2 := \mathbf{E}\Big[\|X_\beta(1) - m_\beta\|^2\Big] \le 2 . \tag{1.12}$$

We will need the following local limit theorem that is uniform in small β.

Lemma 1.4. *There exists* $C = C(d)$ *and* $0 < b_2 = b_2(\mathbf{p}^1) < b_1$ *such that the following hold.*
(i) There exists $n_1 = n_1(\mathbf{p}^1)$ *such that for all* $y \in \mathbb{Z}^d$ *we have*

$$\mathbf{P}(S_\beta(n) = y) \le \frac{2C}{D_\beta^d n^{d/2}} , \tag{1.13}$$

when $n \ge n_1$, $\|\beta\| \le b_2$.
(ii) For any $0 < \epsilon < 1$ *and* $0 < L < \infty$ *there exists* $n_2 = n_2(\mathbf{p}^1, \epsilon, L)$ *such*

that for all $y \in \mathbb{Z}^d$ *such that* $\|y - nm_\beta\| \le L\sqrt{n}$ *we have*

$$\begin{aligned}\mathbf{P}(S_\beta(n) = y) &\le \frac{C(1+\epsilon)}{D_\beta^d n^{d/2}} e^{-d\|y-nm_\beta\|_\beta^2/(2n)}\,,\\ \mathbf{P}(S_\beta(n) = y) &\ge \frac{C(1-\epsilon)}{D_\beta^d n^{d/2}} e^{-d\|y-nm_\beta\|_\beta^2/(2n)}\,.\end{aligned} \tag{1.14}$$

when $n \ge n_2$, $\|\beta\| \le b_2$.

We assumed above that the walk has period 1. Trivial modifications can be made to handle the case of period 2, and we will not make this explicit in our arguments.

Proof of Lemma 1.4. The lemma can be proved by appealing to a local central limit theorem for lattice distributions [3, Theorem 2.5.2]. Note that the standard proof in [3] can be followed, and this gives uniformity in β. $\square$

Specializing to $\beta = 0$, we denote

$$D := \det(Q)^{1/2d}\,.$$

Observe that with the norm introduced in (1.5), we have

$$\sum_{x:\|x\|\le L} 1 \ge cD^d L^d\,, \tag{1.15}$$

for some constant $c > 0$ and all $L \ge 1$. When $d \ge 3$, the Green function $G(x) := \sum_{n=0}^\infty \mathbf{p}^n(o,x)$ satisfies (see [10, Theorem 4.3.5]):

$$G(x) \le \frac{C(d)}{D^d}\|x\|^{2-d}, \quad \text{when } \|x\| \ge L_1 = L_1(\mathbf{p}^1). \tag{1.16}$$

It follows from Lemma 1.4 that there exist $0 < b_3 = b_3(\mathbf{p}^1) < b_1$, $k_2 = k_2(\mathbf{p}^1)$ and $c = c(d) < 1$ such that for $k \ge k_2$ and $\|\beta\| \le b_3$ we have

$$\mathbf{P}(\|S_\beta(k) - km_\beta\| \le \sqrt{k}) \le c\,. \tag{1.17}$$

We now choose $0 < b_0 = b_0(\mathbf{p}^1) \le \min\{b_2, b_3\}$ so that Lemma 1.4, (1.11), (1.12) and (1.17) all hold when $\|\beta\| \le b_0$. We also choose now $r_0 = r_0(\mathbf{p}^1) > 0$ such that $\|v\| \le r_0$ implies $\|\beta\| \le b_0$ for the the unique β such that $m_\beta = v$. The constants b_0 and r_0 will now be fixed for the remainer of the paper.

We are ready to prove Proposition 1.3.

Proof of Proposition 1.3. Choose $k_1 = k_1(\mathbf{p}^1)$ in such a way that $4/\sqrt{k_1} < r_0$ and $k_1 \ge k_2(\mathbf{p}^1)$ for the constant k_2 of (1.17). We also require that $k_1 \ge n_1$ and $k_1 \ge n_2(\epsilon = 1/2, L = 1)$ for the constants n_1, n_2 from Lemma 1.4. Fix x, n and k, and let β be such that $m_\beta = x/n$. Note that the choice of k_1 and the conditions on x and n imply that $\|\beta\| \le b_0$.

It is easy to check that conditional on $S(n) = x$, the joint distribution of $X(1), \dots, X(n)$ is the same as the joint distribution of $X_\beta(1), \dots, X_\beta(n)$ conditioned on $S_\beta(n) = x$. Consequently, the joint distribution of $X(1) -$

$x/n, \dots, X(n)-x/n$, given $S(n)-x=0$ is the same as the joint distribution of $X_\beta(1)-m_\beta, \dots, X_\beta(n)-m_\beta$, given $S_\beta(n)-nm_\beta=0$. Therefore, since $\mathbf{E}[S(k)\,|\,S(n)=x]=(k/n)x$, we have

$$\begin{aligned}\mathbf{E}\Big[\|S(k)\|^2\,\Big|\,S(n)=x\Big] &= \mathbf{E}\Big[\Big\|S(k)-\frac{k}{n}x+\frac{k}{n}x\Big\|^2\,\Big|\,S(n)=x\Big]\\ &= \mathbf{E}\Big[\Big\|S(k)-\frac{k}{n}x\Big\|^2\,\Big|\,S(n)=x\Big]+\frac{k^2}{n^2}\|x\|^2\,.\end{aligned} \tag{1.18}$$

The second term on the right hand side is at most $16k$, by our assumption on $\|x\|$. The first term on the right hand side of (1.18) equals

$$\mathbf{E}\Big[\|S_\beta(k)-km_\beta\|^2\,\Big|\,S_\beta(n)=nm_\beta\Big]\,.$$

Conditional on $S_\beta(n)=nm_\beta$, the variables $X_\beta(1),\dots,X_\beta(n)$ are exchangeable, and it is easy to use $S_\beta(n)-nm_\beta=0$ (expanding the variance) to check that $X^j_\beta(k_1)$ and $X^j_\beta(k_2)$ are negatively correlated for all $1\le k_1<k_2\le n$ and $j=1,\dots,d$. It follows that

$$\mathbf{E}\Big[\|S_\beta(k)-km_\beta\|^2\,\Big|\,S_\beta(n)=nm_\beta\Big]\le k\mathbf{E}\Big[\|X_\beta(1)-m_\beta\|^2\,\Big|\,S_\beta(n)=nm_\beta\Big]\,.$$

It remains to estimate the conditional expectation on the right hand side. Using Lemma 1.4, this is at most

$$\begin{aligned}\sum_{y\in\mathbb{Z}^d}\|y-m_\beta\|^2\mathbf{p}^1_\beta(o,y)\frac{\mathbf{p}^{n-1}_\beta(y,x)}{\mathbf{p}^n_\beta(o,x)} &\le \sum_{y\in\mathbb{Z}^d}\|y-m_\beta\|^2\mathbf{p}^1_\beta(o,y)\frac{4n^{d/2}}{(n-1)^{d/2}}\\ &\le C\mathbf{E}\Big[\|X_\beta(1)-m_\beta\|^2\Big]\le C\Sigma^2_\beta\,.\end{aligned}$$

By (1.12), we obtain the first statement (1.8) of the proposition.

In order to prove (1.9) it is sufficient, due to the just proven part (i), to show that δ_1 can be chosen such that $\mathbf{P}(\|S(k)\|>\sqrt{k}\,|\,S(n)=x)\ge c'>0$. For this, let c be the constant in (1.17). Observe that if $\|y\|\le\sqrt{k}$, we have

$$\|(x-y)-(n-k)m_\beta\|=\|y-km_\beta\|\le\|y\|+k\frac{\|x\|}{n}\le\sqrt{k}+k\frac{4n}{\sqrt{kn}}=5\sqrt{k}\,.$$

We now use Lemma 1.4(ii) with $\epsilon > 0$ satisfying $c(1+\epsilon)/(1-\epsilon) < (1+c)/2$. We write

$$\begin{aligned}
&\mathbf{P}(\|S(k)\| \le \sqrt{k} \,|\, S(n) = x) \\
&= \sum_{y:\|y\|\le\sqrt{k}} \mathbf{p}_\beta^k(o,y) \frac{\mathbf{p}_\beta^{n-k}(y,x)}{\mathbf{p}_\beta^n(o,x)} \\
&\le \sum_{y:\|y\|\le\sqrt{k}} \mathbf{p}_\beta^k(o,y) \frac{C(1+\epsilon)}{D_\beta^d (n-k)^{d/2}} \frac{D_\beta^d n^{d/2}}{C(1-\epsilon)} \\
&\le c\frac{1+\epsilon}{1-\epsilon}\left(\frac{1}{1-\delta_1}\right)^{d/2} \\
&\le \frac{1+c}{2}\left(\frac{1}{1-\delta_1}\right)^{d/2} .
\end{aligned}$$

We choose $\delta_1 = \delta_1(d)$ so that the right hand side is < 1, and this proves part (ii) of the proposition.

The last statement (1.10) now follows easily. Due to exchangability, and part (i), we have

$$\mathbf{E}\Big[\|S(k+k') - S(k)\|^2 \,\Big|\, S(n) = x\Big] = \mathbf{E}\Big[\|S(k')\|^2 \,\Big|\, S(n) = x\Big] \le Ck' .$$

Hence the statement follows from $\mathbf{P}(\|S(k)\| > \sqrt{k} \,|\, S(n) = x) \ge c' > 0$ proved in part (ii). □

2. **Setting up the induction scheme**

We begin by introducing some useful notation. Given an instance of $\mathcal{T}_{n,m}$, consider some small $\delta > 0$, where we assume that δn is an integer. We write

$$X_i = V_{i\delta n} \qquad i = 0, 1, \dots, \lfloor \delta^{-1} \rfloor ,$$

and write $x_i \in \mathbb{Z}^d$, $i = 0, 1, \dots, \lfloor \delta^{-1} \rfloor$ for the random spatial location of X_i, that is, x_i is the unique vertex satisfying $\Phi(X_i) = (x_i, i\delta n)$. Write $\mathcal{T}_{n,m}(\ell)$ for the subtree of $\mathcal{T}_{n,m}$ emanating from V_ℓ off the backbone (including the vertex V_ℓ).

Fix an integer K and subdivide the backbone into N stretches of length $K\delta n$, and a remaining part of length at least δn and less than $K\delta n + \delta n$. That is, we write $n = NK\delta n + K'\delta n + \tilde{n}$, with $0 \le K' < K$ an integer and $\delta n \le \tilde{n} < 2\delta n$.

We begin with some definitions that are depicted in Figure 1.

Definition 2.1. *For ℓ satisfying $i\delta n \le \ell < (i+1)\delta n$ we say that a backbone vertex V_ℓ has the* unique descendant property *(UDP) if among its descendants at level $(i+1)\delta n$ in $\mathcal{T}_{n,m}(\ell)$ there is a unique one that reaches level $(i+2)\delta n$. For any other vertex V of $\mathcal{T}_{n,m}$ at level $i\delta n$ we say that V has*

UDP if among its descendants at level $(i+1)\delta n$ there is a unique one that reaches level $(i+2)\delta n$.

Definition 2.2. *Given an integer $K \geq 1$, a number $\delta > 0$ such that $K\delta \leq (1/2)$ and an instance of $\mathcal{T}_{n,m}$ we say that a sequence $(i, i+1, \ldots, i+K)$ of length $K+1$ is* K-tree-good *if the following holds:*

(1) *There exists a unique $i\delta n \leq \ell_1 < (i+1)\delta n$ such that $\mathcal{T}_{n,m}(\ell_1)$ reaches height $(i+2)\delta n$. Moreover, this unique ℓ_1 satisfies $(i+1/4)\delta n \leq \ell_1 \leq (i+3/4)\delta n$.*

(2) *V_{ℓ_1} has UDP (i.e. has a unique descendant at level $(i+1)\delta n$ in $\mathcal{T}_{n,m}(\ell_1)$ that survives to level $(i+2)\delta n$). We call the unique descendant $\mathcal{Y}_{i+1}$. For all i' satisfying $i+2 \leq i' \leq i+K$ we inductively define the vertices $\mathcal{Y}_{i'}$ of $\mathcal{T}_{n,m}(\ell_1)$ as follows. We require that $\mathcal{Y}_{i'-1}$ has UDP and call the unique descendant $\mathcal{Y}_{i'}$.*

(3) *There exists a unique $(i+K-1)\delta n \leq \ell_2 < (i+K)\delta n$ such that $\mathcal{T}_{n,m}(\ell_2)$ reaches height $(i+K+1)\delta n$. Moreover, this unique ℓ_2 satisfies $(i+K-3/4)\delta n \leq \ell_2 \leq (i+K-1/4)\delta n$.*

(4) *V_{ℓ_2} has UDP, and we call the unique descendant $\mathcal{X}'_{i+K}$. The vertex $\mathcal{X}'_{i+K}$ has UDP, and we call the unique descendant $\mathcal{X}'_{i+K+1}$. Similarly, $\mathcal{Y}_{i+K}$ has UDP, and we call the unique descendant $\mathcal{Y}_{i+K+1}$.*

Given a K-tree-good sequence $(i, \ldots, i+K)$ we denote by $V^+_{\ell_1}$ (respectively $V^+_{\ell_2}$) the child of V_{ℓ_1} (respectively V_{ℓ_2}) leading to $\mathcal{Y}_{i+1}$ (respectively $\mathcal{X}'_{i+K}$). We further define the spatial locations $y_{i'}$ by $\Phi(\mathcal{Y}_{i'}) = (y_{i'}, i'\delta n)$ for $i+1 \leq i' \leq i+K+1$, and we similarly define x'_{i+K}, x'_{i+K+1}, v_{ℓ_1}, $v^+_{\ell_1}$, v_{ℓ_2}, $v^+_{\ell_2}$.

We will write $U \prec W$ to denote that W is a descendant of U, and write $h(U), h(W)$ for their respective heights in the tree (in particular, $h(W) > h(U)$).

Definition 2.3. *Let $U \prec W$ be two tree vertices and let $u, w \in \mathbb{Z}^d$ be defined by $\Phi(U) = (u, h(U))$ and $\Phi(W) = (w, h(W))$. We say that U and W are* typically-spaced *if $\|w - u\| \leq \sqrt{h(W) - h(U)}$. Denote this event by $\mathcal{TS}(U, W)$.*

Definition 2.4. *We say that a K-tree-good sequence $(i, \ldots, i+K)$ is* K-spatially-good *if the following holds.*

(5)
- *$\mathcal{TS}(X_i, V_{\ell_1})$,*
- *$\mathcal{TS}(V_{\ell_1+1}, X_{i+1})$,*
- *For each $i+1 \leq j \leq i+K-2$ we have $\mathcal{TS}(X_j, X_{j+1})$,*
- *$\mathcal{TS}(X_{i+K-1}, V_{\ell_2})$,*
- *$\mathcal{TS}(V_{\ell_2+1}, X_{i+K})$,*

(6)
- *$\mathcal{TS}(V^+_{\ell_1}, \mathcal{Y}_{i+1})$,*
- *For each $i+1 \leq j \leq i+K-1$ we have $\mathcal{TS}(\mathcal{Y}_j, \mathcal{Y}_{j+1})$,*
- *$\mathcal{TS}(V^+_{\ell_2}, \mathcal{X}'_{i+K})$,*
- *$\|x'_{i+K} - y_{i+K}\| \leq \sqrt{\delta n}$.*

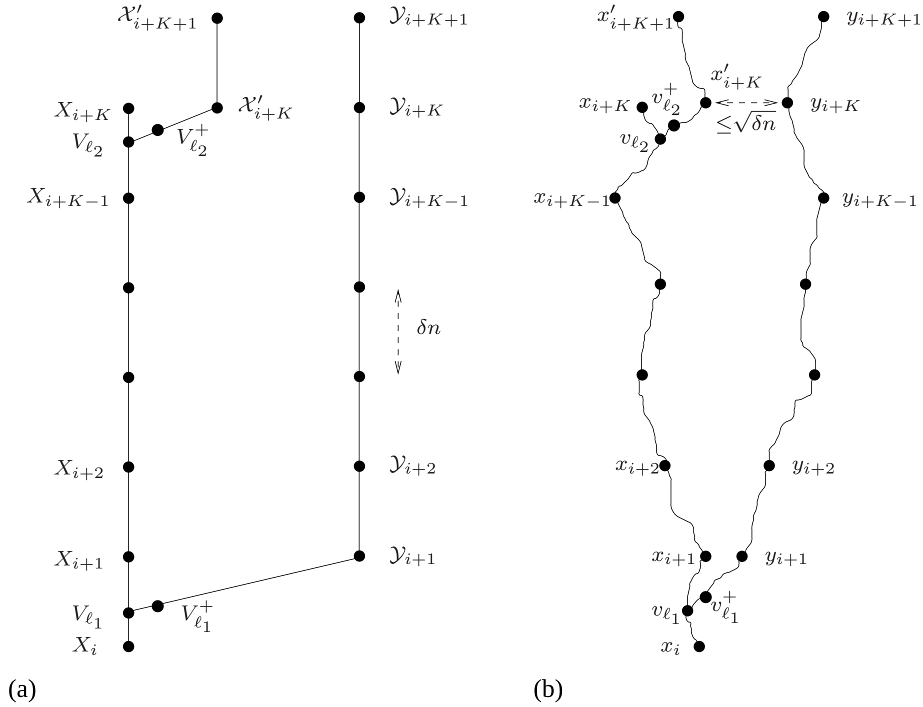

FIGURE 1. (a) Illustration of K-tree-good. (b) Illustration of K-spatially-good. All spatial distances between consecutive vertices are at most $\sqrt{\text{time difference}}$ and the spatial distance between x'_{i+K} and y_{i+K} is at most $\sqrt{\delta n}$.

Definition 2.5. *When a sequence $(i,\dots,i+K)$ is both K-tree-good and K-spatially-good we say that it is K-*good. *Let $\mathcal{A}(i)$ be the event that $(i,\dots,i+K)$ is K-good.*

Next, let $(i,\dots,i+K)$ be a K-good sequence and let U_1, U_2 be two vertices at the same height such that $U_1 \succ \mathcal{X}'_{i+K}$ and $U_2 \succ \mathcal{Y}_{i+K}$. Given these, we write Z_1 for the highest common ancestor of U_1 and $\mathcal{X}'_{i+K+1}$ and Z_2 for the highest common ancestor of U_2 and $\mathcal{Y}_{i+K+1}$ (see Figure 2). Further, we denote by Z_1^+ (respectively Z_2^+) the child of Z_1 (respectively Z_2) leading to U_1 (respectively U_2).

Definition 2.6. *We say that U_1, U_2* intersect-well *if the following conditions hold:*

1. $U_1 \succ \mathcal{X}'_{i+K}$, $U_2 \succ \mathcal{Y}_{i+K}$,
2. $(i+K+(5/6))\delta n \leq h(U_1) = h(U_2) \leq (i+K+1)\delta n$;
3. $(i+K+(1/2))\delta n \leq h(Z_1), h(Z_2) \leq (i+K+(4/6))\delta n$;
4. $\mathcal{TS}(\mathcal{X}'_{i+K}, Z_1)$, $\mathcal{TS}(Z_1^+, U_1)$, $\mathcal{TS}(\mathcal{Y}_{i+K}, Z_2)$, $\mathcal{TS}(Z_2^+, U_2)$;
5. $\Phi(U_1) = \Phi(U_2)$.

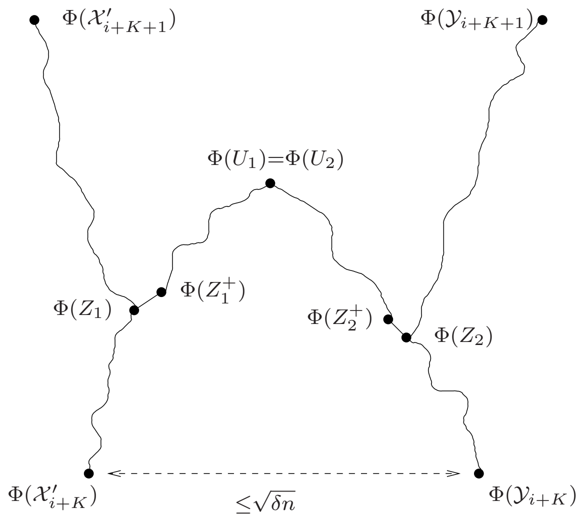

FIGURE 2. The labelling of vertices in the two (potentially) intersecting trees emanating from $\mathcal{X}'_{i+K}$ and $\mathcal{Y}_{i+K}$.

And define the random set $\mathcal{I}$ *by*

$$\mathcal{I} = \{(U_1, U_2) : U_1 \text{ and } U_2 \text{ intersect-well}\} , \tag{2.1}$$

Lastly, we define the event $\mathcal{B}(i, c_0)$ *where* $c_0 > 0$ *is a constant*

$$\mathcal{B}(i, c_0) = \mathcal{A}(i) \cap \left\{|\mathcal{I}| \geq \frac{c_0\sigma^4}{D^d}(\delta n)^{(6-d)/2}\right\}.$$

Our first theorem is that K-good runs $(i, i+1, \ldots, i+K)$ occur with positive density and in each, the probability of seeing many intersections is bounded away from zero. In the two following theorems, the probability measure is the joint distribution of $\mathcal{T}_{n,m}$ and the random mapping Φ.

Theorem 2.1 (Intersections exist)**.** *Assume that* $d = 5$*. There exist constants* $c_0, c_1 > 0$ *and for any* $K \geq 2$ *there exists* $c_2 = c_2(K) > 0$*, and* $n_3 = n_3(\sigma^2, C_3, \mathbf{p}^1, K)$ *such that for any* $0 < \delta < (K+4)^{-1}$*, whenever* $\delta n \geq n_3$ *and* x *satisfies* $\|x\| \leq \sqrt{2n/\delta}$*, we have*

$$\mathbf{P}(\mathcal{A}(i) \mid \Phi(V_n) = (x, n)) \geq c_2 .$$

and

$$\mathbf{P}(\mathcal{B}(i, c_0) \mid \mathcal{A}(i) , \Phi(V_n) = (x, n)) \geq c_1$$

for $i = 0, K, 2K, \ldots, (N-1)K$*.*

Recall the definition of $\gamma(n, x)$ in (1.4). To proceed let us define

$$\gamma(n) = \sup_{x:\|x\|\leq\sqrt{n}} \gamma(n, x) .$$

When all the good events occur, it is immediate by definition and the triangle inequality (1.2) that the expected resistance between X_i and X_{i+K}

is bounded above by $K\gamma(\delta n)$. The following theorem shows that the intersections create a "short-cut" in the electric circuit, allowing us to bound the expected resistance between the two ends of the the run $(i, \ldots, i+K)$ using the parallel law of electric resistance (1.3) essentially by $\frac{3}{4}K\gamma(\delta n)$. This multiplicative constant improvement allows the induction argument to work.

Theorem 2.2 (Analysis of good blocks)**.** *There exists $K_0 < \infty$ and $n_4 = n_4(\sigma^2, C_3, \mathbf{p}^1)$ such that if $K \geq K_0$ and $\delta n \geq n_4$, we have*

$$\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \mid \mathcal{A}(i), \mathcal{B}(i, c_0), \Phi(V_n) = (x, n)\Big] \leq \frac{3K}{4} \max_{1 \leq k \leq \delta n} \gamma(k)$$

for $i = 0, K, 2K, \ldots, (N-1)K$.

To complete the induction step we also need a bound on the expected resistance conditioned on $\mathcal{A}(i)^c \cup \mathcal{B}(i, c_0)^c$. This is rather lengthy, since for each reason that either $\mathcal{A}(i)$ or $\mathcal{B}(i, c_0)$ fail, we provide a different bound on the expected resistance which we eventually collect together at the proof of the induction step.

The role of the small parameter δ will be twofold. On the one hand, in bounding some of the terms arising from the event $\mathcal{A}(i)^c$ we get contributions similar to $(1 + O(\delta))\gamma(\delta n)$, and we will need δ sufficiently small to control these. On the other hand, when $\|x\| \gg \sqrt{n}$, the conditioning on $\Phi(V_n) = (x, n)$ introduces a significant drift for the backbone. In such cases, choosing δ to be $n/\|x\|^2$ ensures that on time scale δn the drift felt by the backbone is only $\delta\|x\| = n/\|x\| = \sqrt{\delta n}$, i.e. the same order as the standard deviation of an unconditioned walk. This is precisely what is required to ensure that good blocks exist on time scale δn, and is the reason for the key condition $\|x\| = O(\sqrt{n/\delta})$ of Theorem 2.1. The choice $\delta = n/\|x\|^2$ explains the extra scaling factor $(\|x\|^2/n)^\alpha$ present in Theorem 1.2. Indeed, heuristically, when $\|x\| \gg \sqrt{n}$, we can bound the total resistance by adding the resistances over each block of length $K\delta n$ and hence expect $\gamma(n, x) \leq (K\delta)^{-1}K\gamma(\delta n) \leq (\delta)^{-1}A(\delta n)^{1-\alpha} = An^{1-\alpha}\left(\frac{\|x\|^2}{n}\right)^\alpha$.

2.1. **Organization.** The proof of Theorem 2.1 is done in Section 3 and the proof of Theorem 2.2 is presented in Section 6. The analysis of the expected resistance when the good events fail to occur is presented in Sections 4 and 5.

3. Existence of intersections

In this section we prove Theorem 2.1. In Section 3.2 we show that K-good runs $(i, \ldots, i+K)$ occur with positive probability, proving the first statement of Theorem 2.1. In Section 3.3 we show that given a K-good run, there are "enough" intersections with positive probability, proving the second statement of Theorem 2.1.

3.1. **Preliminaries.** Recall that $\tilde{p}(k) = (k+1)p(k+1)$ for $k \geq 0$. We denote by $\{\mathcal{N}_n\}_{n\geq 0}$ a branching process with $\mathcal{N}_0 = 1$ and progeny distribution $p(k)$, and by $\{\widetilde{\mathcal{N}}_n\}_{n\geq 0}$ a branching process with $\widetilde{\mathcal{N}}_0 = 1$ and progeny $\tilde{p}(k)$ in the first generation and progeny $p(k)$ afterwards. Note that for all $n \geq 1$ we have $\mathbf{E}\mathcal{N}_n = 1$ and

$$\mathbf{E}\widetilde{\mathcal{N}}_n = \sum_{k\geq 0} k\tilde{p}(k) = \sum_{k\geq 1} k(k-1)p(k) = \mathrm{Var}(\mathcal{N}_1) = \sigma^2 .$$

We denote by $f(s)$ and $\tilde{f}(s) = f'(s)$ the generating functions of p and $\tilde{p}$, respectively. Then the generating functions of $\mathcal{N}_n$ and $\widetilde{\mathcal{N}}_n$ are $f_n(s)$ and $g_n(s) := \tilde{f}(f_{n-1}(s))$, respectively, where $f_n(s)$ is the n-fold composition of f with itself.

We denote by $\theta(n) = f_n(0) = \mathbf{P}(\mathcal{N}_n > 0)$ the survival probability of the branching process up to time n. It is well known [12, 1] that

$$\frac{\theta(n)\sigma^2 n}{2} \to 1 \text{ as } n \to \infty. \tag{3.1}$$

Furthermore, there exists $n_5 = n_5(C_3) < \infty$ such that

$$\frac{1}{\sigma^2 n} \leq \theta(n) \leq \frac{3}{\sigma^2 n}, \qquad n \geq n_5 . \tag{3.2}$$

Moreover, there exists $n_5' = n_5'(C_3)$ such that we have

$$\theta(n) - \theta(m) \geq \frac{1}{2\sigma^2 n}, \quad \text{whenever } m \geq 2n,\ n \geq n_5'. \tag{3.3}$$

Lemma 3.1. *For any $C > 0$ there exists $c' = c'(C) > 0$ and $n_6 = n_6(C, \sigma^2, C_3) < \infty$ such that for all $n \geq n_6$ we have*

$$f_n'\left(1 - \frac{C}{\sigma^2 n}\right) \geq c'$$
$$g_n'\left(1 - \frac{C}{\sigma^2 n}\right) \geq c'\sigma^2 .$$

Proof. We have that

$$f_n'(s) \geq \big(f'(s)\big)^n \geq \big(1 - (1-s)f''(1)\big)^n . \tag{3.4}$$

Indeed, the first inequality follows by appealing to the chain rule and using the fact that $f_n(s) \geq s$ for $s \in [0,1]$ and that f is convex. The second inequality follows from the mean-value theorem together with the fact that f'' is increasing (the coefficients of the Taylor series of f are non-negative by definition). Substituting $s = 1 - C/\sigma^2 n$ gives the first statement (recall that $f''(1) = \sigma^2$). For the second statement, observe that

$$\begin{aligned} g_n'(s) &= \tilde{f}'(f_{n-1}(s))\, f_{n-1}'(s) \geq \tilde{f}'(s)\, \big(f'(s)\big)^{n-1} = f''(s)\ \big(f'(s)\big)^{n-1} \\ &\geq \big(f''(1) - (1-s)f'''(1)\big)\, \big(1 - (1-s)f''(1)\big)^{n-1} . \end{aligned}$$

Substituting $s = 1 - C/\sigma^2 n$ and using that $f'''(1) \leq C_3$ yields the result. $\square$

Lemma 3.2. *There exist $n_7 = n_7(C_3) < \infty$ such that*

$$\frac{1}{2n} \leq \mathbf{P}(\widetilde{\mathcal{N}}_n > 0) \leq \frac{3}{n}, \quad \textit{whenever } n \geq n_7, \tag{3.5}$$

and

$$\frac{1}{4n} \leq \mathbf{P}(\widetilde{\mathcal{N}}_n > 0 \,|\, \widetilde{\mathcal{N}}_m = 0) \leq \frac{6}{n}, \quad \textit{whenever } n \geq n_7,\ m \geq 2n. \tag{3.6}$$

Proof. For the upper bound in (3.5), if the process survives n generations, one of the particles at generation 1 needs to survive $n-1$ generations, so by (3.2) we bound this probability by

$$\sum_{k=1}^{\infty} \tilde{p}(k) k \frac{3}{\sigma^2 n} \leq \frac{3}{n}\,.$$

For the lower bound in (3.5), we write

$$\begin{aligned}
\mathbf{P}(\widetilde{\mathcal{N}}_n > 0) &= 1 - \tilde{f}(f_{n-1}(0)) = 1 - f'(1 - \theta(n-1)) \\
&\geq 1 - f'\left(1 - \frac{1}{\sigma^2(n-1)}\right) \\
&\geq \frac{1}{\sigma^2(n-1)} f''\left(1 - \frac{1}{\sigma^2(n-1)}\right),
\end{aligned} \tag{3.7}$$

where the last inequality is due to the mean-value theorem. As before, $f''(s) \geq (f''(1) - (1-s)f'''(1))$ and $f''(1) = \sigma^2$ and $f'''(1) \leq C_3$ gives the lower bound.

The proof of (3.6) is quite similar. The upper bound follows easily, since by (3.5) when m is large enough we have

$$\mathbf{P}(\tilde{N}_n > 0 \,|\, \widetilde{\mathcal{N}}_m = 0) \leq 2\mathbf{P}(\widetilde{\mathcal{N}}_n > 0)\,.$$

For the lower bound, using (3.3) we write:

$$\begin{aligned}
\mathbf{P}(\widetilde{\mathcal{N}}_n > 0 \,|\, \widetilde{\mathcal{N}}_m = 0) &\geq \mathbf{P}(\widetilde{\mathcal{N}}_n > 0,\, \widetilde{\mathcal{N}}_m = 0) \\
&= \tilde{f}(f_{m-1}(0)) - \tilde{f}(f_{n-1}(0)) \\
&= f'(1 - \theta(m-1)) - f'(1 - \theta(n-1)) \\
&\geq (\theta(n-1) - \theta(m-1))\, f''(1 - \theta(n-1)) \\
&\geq \frac{1}{2\sigma^2 n} f''\left(1 - \frac{c}{\sigma^2 n}\right).
\end{aligned} \tag{3.8}$$

This is now bounded from below as in (3.7). □

3.2. *K*-**good runs occurs.** The proof is broken down into a series of lemmas showing that each of the conditions involved in a run $(i, \ldots, i+K)$ being K-tree-good and K-spatially-good (that is, the conditions in Definitions 2.2 and 2.4) holds with probability bounded away from 0. For $a = 1, \ldots, 6$ let $\mathcal{D}_{(a)}$ denote the event that condition (a) in Definitions 2.2 and 2.4 is satisfied.

We start by analyzing the conditions in Definition 2.2(1)–(4). Recall that these only involve the branching process, hence here the conditioning on $\{\Phi(V_n) = (x, n)\}$ present in Theorem 2.1 is irrelevant. Therefore we omit it in the lemmas below.

Lemma 3.3. *There exists* $c > 0$ *such that we have*

$$\mathbf{P}(\mathcal{D}_{(1)}) \geq c$$

whenever $\delta n \geq n_7$ *.*

Proof. Assume without loss of generality that $i = 0$ (the proof will be the same for any $0 \leq i \leq \delta^{-1}$). For any ℓ satisfying $0 \leq \ell < \delta n$, let $\mathcal{D}_{(1)}(\ell)$ be the event that the random tree attached to V_ℓ, that is $\mathcal{T}_{n,m}(\ell)$, reaches level $2\delta n$. So $\mathcal{D}_{(1)}$ is the event that exactly one of the events $\{\mathcal{D}_{(1)}(\ell)\}$ occurs, and that the index ℓ_1 of that event lies between $(1/4)\delta n$ and $(3/4)\delta n$. The events $\{\mathcal{D}_{(1)}(\ell)\}$ are independent, and due to (3.6) each has probability between $\frac{1}{8\delta n}$ and $\frac{6}{\delta n}$. Hence $\mathbf{P}(\mathcal{D}_{(1)}) \geq c > 0$. □

Lemma 3.4. *There exists* $c = c(K) > 0$ *such that we have*

$$\mathbf{P}(\mathcal{D}_{(2)} \,|\, \mathcal{D}_{(1)},\, \ell_1) \geq c > 0$$

whenever $\delta n \geq \max\{n_5, n_5', 4n_6(3, \sigma^2, C_3), n_7\}$.

Proof. Again we assume that $i = 0$ (the reader will notice that we only use the fact that $m - i\delta n \geq n$). The probability that V_{ℓ_1} has UDP given $\mathcal{D}_{(1)}$ equals

$$\frac{\sum_{k\geq 1} \mathbf{P}(\widetilde{\mathcal{N}}_{\delta n-\ell_1} = k)\, k\, (\theta(\delta n) - \theta(m - \delta n))\, (1 - \theta(\delta n))^{k-1}}{\mathbf{P}(\widetilde{\mathcal{N}}_{2\delta n-\ell_1} > 0,\, \widetilde{\mathcal{N}}_{m-\ell_1} = 0)},$$

since $\mathcal{T}_{n,m}(\ell_1)$ is now conditioned to survive $2\delta n - \ell_1$ generations, but that the $m - \ell_1$-th generation died out. Hence

$$\mathbf{P}(V_{\ell_1} \text{ has UDP} \,|\, \mathcal{D}_{(1)},\, \ell_1) \geq \frac{\theta(\delta n) - \theta(m - \delta n)}{\mathbf{P}(\widetilde{\mathcal{N}}_{2\delta n-\ell_1} > 0)} g'_{\delta n-\ell_1}(1 - \theta(\delta n))\,. \tag{3.9}$$

Due to (3.3) and Lemmas 3.2 and 3.1, the right hand side of (3.9) is at least a universal constant $c' > 0$.

Now, conditioned on V_{ℓ_1} having UDP, the descendant tree emanating from $\mathcal{Y}_1$ is a critical tree conditioned to survive δn generations but not $m - \delta n$ generations. So the conditional probability that $\mathcal{Y}_1$ has UDP equals

$$\frac{\sum_{k\geq 1} \mathbf{P}(\mathcal{N}_{\delta n} = k)\, k\, (\theta(\delta n) - \theta(m - \delta n))\, (1 - \theta(\delta n))^{k-1}}{\mathbf{P}(\mathcal{N}_{\delta n} > 0,\, \mathcal{N}_{m-\delta n} = 0)},$$

and similarly this is bounded below by c'. Iterating this argument over $\mathcal{Y}_2, \mathcal{Y}_3, \ldots \mathcal{Y}_{K-1}$ gives a probability of at least $c(K) = c'^K$, as required. □

Lemma 3.5. *There exists $c > 0$ such that*

$$\mathbf{P}(\mathcal{D}_{(3)} \,|\, \mathcal{D}_{(1)},\, \ell_1,\, \mathcal{D}_{(2)}) = \mathbf{P}(\mathcal{D}_{(3)}) \geq c$$

whenever $\delta n \geq n_7$.

Proof. The proof is the same as the proof of Lemma 3.3. □

Lemma 3.6. *There exists $c > 0$ such that*

$$\mathbf{P}(\mathcal{D}_{(4)} \,|\, \mathcal{D}_{(1)},\, \ell_1,\, \mathcal{D}_{(2)},\, \mathcal{D}_{(3)},\, \ell_2) \geq c$$

whenever $\delta n \geq \max\{n_5, n_5', 4n_6(3, \sigma^2, C_3), n_7\}$.

Proof. This is proved almost identically to Lemma 3.4. □

We next show that the conditions in Definition 2.4(5)–(6) also hold with probability bounded away from 0.

Lemma 3.7. *There exists $c = c(K) > 0$ and $n_8 = n_8(\mathbf{p}^1, K)$ such that whenever $0 < \delta < 1/(K+4)$, $\delta n \geq n_8$ and $\|x\| \leq \sqrt{2n/\delta}$, we have*

$$\mathbf{P}\big(\mathcal{D}_{(5)}, \mathcal{D}_{(6)} \,|\, \mathcal{D}_{(1)},\, \ell_1,\, \mathcal{D}_{(2)}, \mathcal{D}_{(3)},\, \ell_2,\, \mathcal{D}_{(4)},\, \Phi(V_n) = (x, n)\big) \geq c\,. \tag{3.10}$$

Proof. Let us condition on the entire branching process tree $\mathcal{T}_{n,m}$ in which $\mathcal{D}_{(1)}$–$\mathcal{D}_{(4)}$ hold. It will be convenient to consider the event $\mathcal{D}'_{(5)} \subset \mathcal{D}_{(5)}$ where we replace the requirements in Definition 2.4,(5) by

(i) $\|x_i - v_{\ell_1}\| \leq (1/2)\sqrt{\ell_1 - i\delta n}$,
(ii) $\|v_{\ell_1+1} - x_{i+1}\| \leq (1/2)\sqrt{i\delta n - \ell_1 - 1}$,
(iii) For each $i+1 \leq j \leq i+K-2$ we have $\|x_j - x_{j+1}\| \leq (1/2)\sqrt{\delta n}$,
(iv) $\|x_{i+K-1} - v_{\ell_2}\| \leq (1/2)\sqrt{\ell_2 - (i+K-1)\delta n}$,
(v) $\|x_{i+K} - v_{\ell_2+1}\| \leq (1/2)\sqrt{(i+K)\delta n - \ell_2 - 1}$.
(vi) $\|v_{\ell_1} - v_{\ell_1+1}\|$, $\|v_{\ell_2} - v_{\ell_2+1}\| \leq \sqrt{2}$

We will show that

$$\mathbf{P}(\mathcal{D}'_{(5)}, \mathcal{D}_{(6)} | \mathcal{T}_{n,m}) \geq c\,, \tag{3.11}$$

and that

$$\mathbf{P}(\Phi(V_n) = (x, n) \,|\, \mathcal{D}'_{(5)}, \mathcal{D}_{(6)}, \mathcal{T}_{n,m}) \geq c\mathbf{P}(\Phi(V_n) = (x, n) | \mathcal{T}_{n,m})\,, \tag{3.12}$$

which will conclude our proof. To prove (3.11) we first note that the events of $\mathcal{D}'_{(5)}$ are all independent and each occurs with probability bounded below by a constant, by (1.7) and (1.6). Conditioned on $x_i, x_{\ell_1}, x_{\ell_1+1}, x_{i+1}, \ldots, x_{i+K}$ that satisfy $\mathcal{D}'_{(5)}$, the event $\mathcal{D}_{(6)}$ has probability at least $c = c(K) > 0$, indeed, because of the factors 1/2 in the definition of $\mathcal{D}'_{(5)}$, repeated application of the central limit theorem yields that the displacement requirements in $\mathcal{D}_{(6)}$ can be satisfied.

To prove (3.12) we condition on the value of $z = x_{i+K} - x_i$. Choose n_8 large enough so that the conditions $\|x\| \leq \sqrt{2n/\delta}$ and $\delta n \geq n_8$ imply $\|x/n\| \leq r_0$ (where r_0 is the constant chosen in Section 1.5). Let β be such that $m_\beta = x/n$.

Observe that

$$\|z - K\delta n m_\beta\| \le (K+4)\sqrt{\delta n} + K\delta n\|x\|/n \le 5\sqrt{K}\sqrt{K\delta n}\,,$$

where we used $\|x\| \le \sqrt{2n/\delta}$ in the last inequality. We now also require $n_8 \ge n_2(\mathbf{p}^1, \epsilon = 1/2, L = 5\sqrt{K})$, where n_2 is the constant in Lemma 1.4. We have

$$\begin{aligned}\mathbf{P}\Big(\Phi(V_n) = (x,n)\,\Big|\, \mathcal{D}'_{(5)},\, \mathcal{D}_{(6)},\, \mathcal{T}_{n,m},\, z\Big) &= \mathbf{P}\Big(S(n - K\delta n) = x - z\Big) \\ &= \mathbf{p}^{n-K\delta n}(z,x) \\ &= \frac{Z_\beta^{n-K\delta n}}{e^{\beta\cdot(x-z)}}\mathbf{p}_\beta^{n-K\delta n}(z,x)\end{aligned}$$

and

$$\begin{aligned}\mathbf{P}\Big(\Phi(V_n) = (x,n)\,\Big|\, \mathcal{T}_{n,m}\Big) &= \mathbf{P}(S(n) = x) \\ &= \mathbf{p}^n(0,x) \\ &= \frac{Z_\beta^n}{e^{\beta\cdot x}\mathbf{p}_\beta^n(0,x)}.\end{aligned}$$

Lemma 1.4(ii) implies that

$$\begin{aligned}\frac{\mathbf{p}^{n-K\delta n}(z,x)}{\mathbf{p}^n(o,x)} &= \frac{e^{\beta\cdot z}}{Z_\beta^{K\delta n}}\frac{\mathbf{p}_\beta^{n-K\delta n}(z,x)}{\mathbf{p}_\beta^n(o,x)} \\ &= \frac{\mathbf{p}_\beta^{K\delta n}(o,z)}{\mathbf{p}^{K\delta n}(o,z)}\frac{\mathbf{p}_\beta^{n-K\delta n}(z,x)}{\mathbf{p}_\beta^n(o,x)} \\ &\ge \frac{c(K)}{D_\beta^d(K\delta n)^{d/2}}\frac{D^d(K\delta n)^{d/2}}{2C}\frac{c(K)}{D_\beta^d(n-K\delta n)^{d/2}}\frac{D_\beta^d n^{d/2}}{2C} \\ &\ge c'(K).\end{aligned}$$

This in turn implies (3.12). □

3.3. **Abundant intersections.** We proceed with proving the second part of Theorem 2.1. To ease the presentation of this calculation let $\mathcal{T}_1$ and $\mathcal{T}_2$ be independent random trees distributed as $\mathcal{T}_{\delta n, 2\delta n}$ and rooted at ρ_1, ρ_2, respectively. Let Φ_1 and Φ_2 be independent random walk mappings of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, into $\mathbb{Z}^d \times \mathbb{Z}_+$ such that $\Phi_1(\rho_1) = (o,0)$ and $\Phi_2(\rho_2) = (x,0)$. Then on the event $\mathcal{A}(i) \cap \{y_{i+K} - x'_{i+K} = x\}$, the random variable $|\mathcal{I}|$ introduced in (2.1) has the same distribution as the random variable (also denoted $|\mathcal{I}|$ here):

$$|\mathcal{I}| = \sum_{U_1\in\mathcal{T}_1, U_2\in\mathcal{T}_2} \mathbf{1}_{(U_1,U_2)\text{ intersect-well}}\,.$$

Here we have tacitly adapted the definition of "intersect-well" to the present setting, by replacing $\mathcal{X}'_{i+K}$ by ρ_1 and $\mathcal{Y}_{i+K}$ by ρ_2. Our goal in this section is to show that when $d = 5$ we have $|\mathcal{I}| \geq c\sigma^4 D^{-5}(\delta n)^{1/2}$ with positive probability.

Theorem 3.8. *Assume that $d = 5$ and $\|x\| \leq \sqrt{\delta n}$. There exist constants $C < \infty$, $c > 0$ and $n_9 = n_9(\sigma^2, C_3, \mathbf{p}^1) < \infty$ such that for $\delta n \geq n_9$ we have*

$$\mathbf{E}|\mathcal{I}| \geq \frac{c\sigma^4}{D^5}\sqrt{\delta n}\,,$$

and

$$\mathbf{E}|\mathcal{I}|^2 \leq \frac{C\sigma^8}{D^{10}}\delta n\,. \tag{3.13}$$

Recall that for a tree vertex V we write $h(V)$ for its distance from the root. Also recall the vertices Z_1, Z_1^+, Z_2, Z_2^+ introduced before Definition 2.6, and the constant n_2 of Lemma 1.4(ii).

Lemma 3.9. *Given instances of $\mathcal{T}_1$ and $\mathcal{T}_2$, let $U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2$ be vertices both at height $(5/6)\delta n \leq h(U_1) = h(U_2) \leq \delta n$, and such that $(1/2)\delta n \leq h(Z_1), h(Z_2) \leq (4/6)\delta n$. There exists $c = c(d) > 0$ such that whenever $\delta n \geq 6n_2(\mathbf{p}^1, \epsilon = 1/2, L = 1)$ and $\|x\| \leq \sqrt{\delta n}$ we have*

$$\mathbf{P}\big((U_1, U_2) \text{ intersect-well} \,\big|\, \mathcal{T}_1, \mathcal{T}_2\big) \geq \frac{c}{D^d(\delta n)^{d/2}}\,.$$

Proof. Denote the spatial locations of Z_1, Z_1^+, Z_2, Z_2^+ by z_1, z_1^+, z_2, z_2^+, and denote the common spatial location of U_1 and U_2 by u. Let us choose the spatial locations so that the inequalities

$$\begin{aligned} \|z_1\| &\leq \sqrt{(1/2)\delta n}\,, & \|z_2 - x\| &\leq \sqrt{(1/2)\delta n}\,, \\ \|z_1^+ - z_1\| &\leq \sqrt{2}\,, & \|z_2^+ - z_2\| &\leq \sqrt{2}\,, \\ \|z_1^+ - u\| &\leq \sqrt{(1/6)\delta n}\,, & \|z_2^+ - u\| &\leq \sqrt{(1/6)\delta n}\,, \end{aligned}$$

are satisfied — this guarantees that the required events $\mathcal{TS}(\cdot,\cdot)$ all occur. Fix the displacements $z_1^+ - z_1$ and $z_2^+ - z_2$. Since $\sqrt{1/2} + \sqrt{1/6} > 1/2$, there are $\geq cD^{3d}(\delta n)^{3d/2}$ choices for (z_1, u, z_2) satisfying the requirements above. Due to Lemma 1.4, each choice has probability at least $c(D^{-d}(\delta n)^{-d/2})^4$ occurring. Combined with (1.6) to handle the displacements $z_1^+ - z_1$ and $z_2^+ - z_2$, this proves the statement of the lemma. □

Lemma 3.10. *Assume that $d = 5$ and $\|x\| \leq \sqrt{\delta n}$. We have*

$$\mathbf{E}|\mathcal{I}| \geq \frac{c\sigma^4}{D^d}(\delta n)^{(6-d)/2}\,. \tag{3.14}$$

whenever $\delta n \geq \max\{n_2(\mathbf{p}^1, \epsilon = 1/2, L = 1), n_5, 6n_6\}$.

Proof. By Lemma 3.9 we have

$$\mathbf{E}|\mathcal{I}| \geq \frac{c}{D^d(\delta n)^{d/2}} \sum_{h=5\delta n/6}^{\delta n} \sum_{k_1,k_2=\delta n/2}^{4\delta n/6} \mathbf{E}\mathcal{L}(h, k_1)\mathbf{E}\mathcal{L}(h, k_2) \tag{3.15}$$

where $\mathcal{L}(h,k)$ counts the number of $U_1 \in \mathcal{T}_1$ at level h such that Z_1 is at level k. Note that since Z_1 is a backbone vertex, we have that

$$\mathbf{E}\mathcal{L}_{h,k_1} = \mathbf{E}\Big[\widetilde{\mathcal{N}}_{h-k_1} \,\Big|\, \widetilde{\mathcal{N}}_{2\delta n-k_1} = 0\Big] .$$

We have that

$$\begin{aligned} \mathbf{E}\Big[\widetilde{\mathcal{N}}_{h-k_1} \,\Big|\, \widetilde{\mathcal{N}}_{2\delta n-k_1} = 0\Big] &= (1-\theta(2\delta n - k_1))^{-1} g'_{h-k_1}(1-\theta(2\delta n - h)) \\ &\geq c\sigma^2 , \end{aligned}$$

by Lemma 3.2 and Lemma 3.1. Summing this estimate in (3.15) concludes the proof. □

The remainder of this section is devoted to the proof of the second moment estimate in Theorem 3.8. Given numbers h_u, h_w, k_1 satisfying

$$\delta n/2 \leq k_1 \leq h_u, h_w \leq \delta n ,$$

we write $\mathcal{L}(h_u, h_w, k_1)$ for the variable counting the number of pairs of tree vertices U, W such that their highest common ancestor in the tree is at level k_1.

Lemma 3.11. *We have*

$$\begin{aligned} &\mathbf{E}_{\mathcal{T}_{\delta n, 2\delta n}} \mathcal{L}(h_u, h_w, k_1) \\ &\qquad \leq (C_3 + 2\sigma^4 \delta n)\mathbf{1}_{\{h_u > k_1, h_w > k_1\}} + (1 + 2\sigma^2\delta n)\mathbf{1}_{\{h_u = k_1 \text{ or } h_w = k_1\}} . \end{aligned}$$

Proof. Let $\mathcal{T}_{\delta n,\infty}$ be a random tree obtained similarly to $\mathcal{T}_{\delta n, 2\delta n}$ dropping the requirement that the critical trees hanging on the backbone are conditioned not to reach level $2\delta n$. By the FKG inequality [5, 4] we have

$$\mathbf{E}_{\mathcal{T}_{\delta n, 2\delta n}} \mathcal{L}(h_u, h_w, k_1) \leq \mathbf{E}_{\mathcal{T}_{\delta n,\infty}} \mathcal{L}(h_u, h_w, k_1) ,$$

indeed, the measure $\mathcal{T}_{\delta n, 2\delta n}$ is obtained from $\mathcal{T}_{\delta n,\infty}$ by conditioning on a monotone decreasing event in a product measure (all the independent progeny random variables) and the random variable $\mathcal{L}$ is monotone increasing. From here we will always calculate with respect to $\mathcal{T}_{\delta n,\infty}$ and we drop the corresponding subscript.

For two vertices U, W at heights h_u, h_w we write S for their highest common ancestor at height k_1. There is a slight difference in the calculation depending on whether S is in the backbone of $\mathcal{T}_{\delta n,\infty}$ or not. Write $\mathcal{L}^1(h_u, h_w, k_1)$ for the number of U, W such that S is not on the backbone and $\mathcal{L}^2(h_u, h_w, k_1)$ when S is on the backbone. We first estimate $\mathbf{E}\mathcal{L}^1$. When $h_u > k_1$ and $h_w > k_1$, the expected number of pairs U, W emanating from a fixed S at height k_1 is at most

$$\sum_{k=2}^{\infty} \mathbf{p}(k)\, k\,(k-1) \mathbf{E}\mathcal{N}_{h_u - k_1 - 1}\, \mathbf{E}\mathcal{N}_{h_w - k_1 - 1} = \sigma^2 .$$

When either $h_u = k_1$ or $h_w = k_1$ (that is, either U or W equal S) the expected number of such pairs is at most 1. By summing over the backbone vertex from which S emanates we have that

$$\mathbf{E}\mathcal{L}^1(h_u, h_w, k_1) \le \sigma^4 \delta n \mathbf{1}_{\{h_u > k_1, h_w > k_1\}} + \sigma^2 \delta n \mathbf{1}_{\{h_u = k_1 \text{ or } h_w = k_1\}} \,.$$

To estimate $\mathbf{E}\mathcal{L}^2$ we assume now that S is the unique vertex on the backbone at height k_1, and when $h_u > k_1$ and $h_w > k_1$ the expected number of U, W in $\mathcal{T}_{\delta n, \infty}(k_1)$ is

$$\sum_{k=2}^{\infty} \tilde{\mathbf{p}}(k)\, k\,(k-1) \mathbf{E}\mathcal{N}_{h_u - k_1 - 1}\, \mathbf{E}\mathcal{N}_{h_w - k_1 - 1} \le C_3 \,.$$

The expected number of U, W such that $U \in \mathcal{T}_{\delta n, \infty}(k_1)$ but W emanates from some other backbone vertex at height $> k_1$ is at most $\sigma^4 \delta n$. Similarly, the expected number of U, W in which $h_u = k_1$ (and so $U = S$) is at most $\sigma^2 \delta n$. Putting these together gives

$$\begin{aligned} &\mathbf{E}\mathcal{L}^1(h_u, h_w, k_1) \\ &\qquad \le (C_3 + \sigma^4 \delta n) \mathbf{1}_{\{h_u > k_1, h_w > k_1\}} + (1 + \sigma^2 \delta n) \mathbf{1}_{\{h_u = k_1 \text{ or } h_w = k_1\}} \,. \end{aligned}$$

□

Recall the constant $n_1(\mathbf{p}^1)$ of Lemma 1.4 and the constant $L_1(\mathbf{p}^1)$ of (1.16). A key "diagrammatic estimate" needed for the second moment bound is provided by the following Lemma. See Figure 3(a).

Lemma 3.12. *Suppose $d \ge 3$. There are constants $C = C(d) > 0$ and $C_2 = C_2(\mathbf{p}^1)$ such that*

$$\sum_{h : k_1 \vee k_2 \le h \le \delta n} \mathbf{p}^{2h - k_1 - k_2}(z_1, z_2) \le \frac{C}{D^d} f(k_1, k_2, z_1, z_2) \,,$$

where

$$f(k_1, k_2, z_1, z_2) := \begin{cases} |k_1 - k_2|^{(2-d)/2} & \text{if } \|z_1 - z_2\| \le |k_1 - k_2|^{1/2} \\ & \text{and } |k_1 - k_2| \ge n_1; \\[2ex] C_2 & \text{if } \|z_1 - z_2\| \le |k_1 - k_2|^{1/2} < \sqrt{n_1}; \\[2ex] \|z_1 - z_2\|^{2-d} & \text{if } \|z_1 - z_2\| > |k_1 - k_2|^{1/2} \\ & \text{and } \|z_1 - z_2\| \ge L_1; \\[2ex] C_2 & \text{if } |k_1 - k_2|^{1/2} < \|z_1 - z_2\| < L_1. \end{cases}$$

Proof. Suppose first we are in the case $\|z_1 - z_2\| \le |k_1 - k_2|^{1/2}$. Then for all $h \ge k_1 \vee k_2$ we have $2h - k_1 - k_2 \ge |k_1 - k_2| \ge \|z_1 - z_2\|^2$. Hence due to

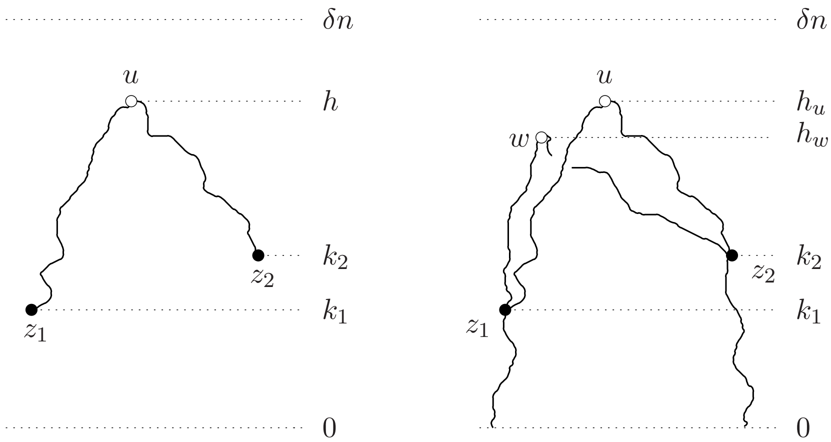

FIGURE 3. (a) Illustration of the quantity bounded in Lemma 3.12. The curves represent random walk transition probabilities between the indicated space-time points. The expression is summed over u to obtain $\mathbf{p}^{2h-k_1-k_2}(z_1, z_2)$ and then summed over h. When $\|z_1 - z_2\| > |k_1 - k_2|^{1/2}$, we get the Green's function decay from the spatial separation $\|z_1 - z_2\|$. When $\|z_1 - z_2\| \leq |k_1 - k_2|^{1/2}$ we get decay from the time separation $|k_1 - k_2|$. (b) Illustration of the quantity Y_1 that contains "two copies" of f.

Lemma 1.4, in the case when $|k_1 - k_2|$ is large enough, we have

$$\begin{aligned}\sum_{h:k_1 \vee k_2 \leq h \leq \delta n} \mathbf{p}^{2h-k_1-k_2}(z_1, z_2) &\leq \sum_{k=|k_1-k_2|}^{\infty} \mathbf{p}^k(z_1, z_2) \\ &\leq \frac{C}{D^d} \sum_{k=|k_1-k_2|}^{\infty} k^{-d/2} \\ &\leq \frac{C}{D^d} |k_1 - k_2|^{(2-d)/2}.\end{aligned}$$

When $|k_1 - k_2|$ is not large, the bound follows trivially.

Suppose now we are in the other case $\|z_1 - z_2\| > |k_1 - k_2|^{1/2}$. Then due to (1.16), in the case when $\|z_1 - z_2\|$ is large enough, we have

$$\sum_{h:k_1 \vee k_2 \leq h \leq \delta n} \mathbf{p}^{2h-k_1-k_2}(z_1, z_2) \leq \sum_{k=0}^{\infty} \mathbf{p}^k(z_1, z_2) \leq \frac{C}{D^d} \|z_1 - z_2\|^{2-d}.$$

The bound is trivial in the case when $\|z_1 - z_2\|$ is not large. □

Proof of Theorem 3.8. The lower bound on the first moment is Lemma 3.10 (we require that $n_9 \geq \max\{6n_2(\mathbf{p}^1, \epsilon = 1/2, L = 1), n_5, 6n_6\}$). We are left to prove the upper bound on the second moment. First we drop the requirements of "typically spaced" from the definition of $\mathcal{I}$. This gives that

$$\mathbf{E}|\mathcal{I}|^2 \leq \sum_{k_1,k_2=\delta n/2}^{\delta n} \sum_{h_u,h_w=k_1\vee k_2}^{\delta n} \mathbf{E}\mathcal{L}(h_u,h_w,k_1)\mathbf{E}\mathcal{L}(h_u,h_w,k_2) \times p(h_u,h_w,k_1,k_2)\,, \tag{3.16}$$

where $p(h_u,h_w,k_1,k_2)$ is the probability that $\Phi_1(U_1) = \Phi_2(U_2)$ and $\Phi_1(W_1) = \Phi_2(W_2)$ where U_1, U_2, W_1, W_2 are any tree vertices satisfying that the highest common ancestor of U_1 and W_1 is at height k_1 and the highest common ancestor of U_2 and W_2 is at height k_2 and $h(U_1) = h(U_2) = h_u$ and $h(W_1) = h(W_2) = h_w$. Note that this probability only depends on the corresponding heights and not on the vertices. We have that

$$p(h_u,h_w,k_1,k_2) = \sum_{z_1,z_2\in\mathbb{Z}^d} \sum_{u,w\in\mathbb{Z}^d} \mathbf{p}^{k_1}(o,z_1)\mathbf{p}^{h_u-k_1}(z_1,u)\mathbf{p}^{h_w-k_1}(z_1,w) \times \mathbf{p}^{k_2}(x,z_2)\mathbf{p}^{h_u-k_2}(z_2,u)\mathbf{p}^{h_w-k_2}(z_2,w)\,. \tag{3.17}$$

We can perform the summations over u, w yielding the expression

$$\sum_{z_1,z_2\in\mathbb{Z}^d} \mathbf{p}^{k_1}(o,z_1)\mathbf{p}^{2h_u-k_1-k_2}(z_1,z_2)\mathbf{p}^{2h_w-k_1-k_2}(z_1,z_2)\mathbf{p}^{k_2}(x,z_2)\,. \tag{3.18}$$

Using Lemmas 3.11 and 3.12 we sum (3.16) over $h_u, h_w > k_1 \vee k_2$ and we get a bound of

$$Y_1 = \frac{C(\sigma^4\delta n)^2}{D^{2d}} \sum_{k_1,k_2=\delta n/2}^{\delta n} \sum_{z_1,z_2} f(k_1,k_2,z_1,z_2)^2\mathbf{p}^{k_1}(o,z_1)\mathbf{p}^{k_2}(x,z_2)\,;$$

see Figure 3(b). Similarly, we sum over $h_u > k_1 \vee k_2$ and $h_w = k_1 \vee k_2$ and when the roles of h_u and h_w exchanged, getting a bound of

$$Y_2 = \frac{C(\sigma^3\delta n)^2}{D^d} \sum_{k_1,k_2=\delta n/2}^{\delta n} \sum_{z_1,z_2} f(k_1,k_2,z_1,z_2)\mathbf{p}^{|k_1-k_2|}(z_1,z_2)\mathbf{p}^{k_1}(o,z_1)\mathbf{p}^{k_2}(x,z_2)\,.$$

And finally our third bound is when $h_u = h_w = k_1 \vee k_2$ giving

$$Y_3 = C(\sigma^2\delta n)^2 \sum_{k_1,k_2=\delta n/2}^{\delta n} \sum_{z_1,z_2} \mathbf{p}^{k_1}(o,z_1)\mathbf{p}^{|k_1-k_2|}(z_1,z_2)\mathbf{p}^{|k_1-k_2|}(z_1,z_2)\mathbf{p}^{k_2}(x,z_2)\,,$$

so that $\mathbf{E}|\mathcal{I}|^2 \leq Y_1 + Y_2 + Y_3$. We start with bounding Y_1. We split the summation over z_1, z_2 into two parts:

(I) $\|z_2 - z_1\| \leq |k_1 - k_2|^{1/2}$;
(II) $\|z_2 - z_1\| > |k_1 - k_2|^{1/2}$.

For the bounds we are going to require $n_9 \geq 2n_1$. We first bound case (I), and initially restrict to $|k_1 - k_2| \geq n_1$ where n_1 is from Lemma 1.4. Using

Lemma 1.4 and $k_1 \geq \delta n/2 \geq n_9/2 \geq n_1$ in the first step, the sum over z_1, z_2 in Y_1 is at most

$$\begin{aligned}
&\sum_{z_2 \in \mathbb{Z}^d} \sum_{z_1 : \|z_2 - z_1\| \leq |k_1 - k_2|^{1/2}} \mathbf{p}^{k_1}(0, z_1)\mathbf{p}^{k_2}(x, z_2)|k_1 - k_2|^{2-d} \\
&\qquad \leq \frac{C}{D^d(\delta n)^{d/2}} \sum_{z_2 \in \mathbb{Z}^d} \sum_{z_1 : \|z_2 - z_1\| \leq |k_1 - k_2|^{1/2}} \mathbf{p}^{k_2}(x, z_2)|k_1 - k_2|^{2-d} \qquad (3.19) \\
&\qquad \leq \frac{C}{(\delta n)^{d/2}} \sum_{z_2 \in \mathbb{Z}^d} \mathbf{p}^{k_2}(x, z_2)|k_1 - k_2|^{2-\frac{d}{2}} = \frac{C|k_1 - k_2|^{2-\frac{d}{2}}}{(\delta n)^{d/2}} .
\end{aligned}$$

Now we sum this over k_1, k_2 and get a bound of $C(\delta n)^{4-d}$. Similarly, when summing over k_1, k_2 satisfying $|k_1 - k_2| \leq n_1$ we get a bound of $C(\delta n)^{1-d/2}$ which is negligible since $d = 5$. Putting all these together gives a contribution to Y_1 from case 1 that is of order $D^{-2d}\sigma^8(\delta n)^{6-d}$.

In case (II) we initially restrict to $\|z_1 - z_2\| \geq L_1$. We have

$$\begin{aligned}
&\sum_{z_2 \in \mathbb{Z}^d} \sum_{z_1 : \|z_1 - z_2\| > |k_1 - k_2|^{1/2}} \mathbf{p}^{k_1}(0, z_1)\mathbf{p}^{k_2}(x, z_2)\|z_1 - z_2\|^{4-2d} \\
&\qquad \leq \frac{C}{D^d(\delta n)^{d/2}} \sum_{z_2 \in \mathbb{Z}^d} \sum_{z_1 : \|z_1 - z_2\| > |k_1 - k_2|^{1/2}} \mathbf{p}^{k_2}(x, z_2)\|z_1 - z_2\|^{4-2d} \\
&\qquad \leq \frac{C}{(\delta n)^{d/2}} \sum_{z_2 \in \mathbb{Z}^d} \mathbf{p}^{k_2}(x, z_2)|k_1 - k_2|^{(4-d)/2} = \frac{C|k_1 - k_2|^{2-\frac{d}{2}}}{(\delta n)^{d/2}} .
\end{aligned}$$

The case $\|z_1 - z_2\| \leq L_1$ is dealt with similarly, and all together we get that n_9 can be chosen in such a way that

$$Y_1 \leq CD^{-2d}\sigma^8(\delta n)^{6-d} .$$

Very similar calculations yield that

$$Y_2 \leq CD^{-2d}\sigma^6(\delta n)^{3-d/2} , \qquad Y_3 \leq CD^{-2d}\sigma^4(\delta n)^{3-d/2} ,$$

concluding the proof. □

Proof of Theorem 2.1. The first part of the theorem is just a combination of Lemmas 3.3, 3.4, 3.5, 3.6 and 3.7, where we take

$$n_3 = \max\{n_5, n_5', 6n_6(3, \sigma^2, C_3), n_7, n_8(\mathbf{p}^1, K), n_9(\sigma^2, C_3, \mathbf{p}^1)\}.$$

For the second part of the theorem we now choose $c_0 = c/2$, where c is the constant in the lower bound on the first moment in Theorem 3.8. Then the second statement of Theorem 2.1 follows immediately from Theorem 3.8 together with the inequality

$$\mathbf{P}\Big(V \geq \frac{1}{2}\mathbf{E}V\Big) \geq \frac{(\mathbf{E}V)^2}{4\mathbf{E}V^2} ,$$

valid for any non-negative random variable V. □

4. Analysis of tree bad blocks

In this section we bound the expected resistance between $\Phi(X_i)$ and $\Phi(X_{i+K})$ conditioned on one of the good events in Definition 2.2 not occurring. We will give a bound in terms of the following quantity, which later we will bound inductively. For any $k \le n$ define

$$\bar{\gamma}(k;(x,n)) = \sum_{y \in \mathbb{Z}^d} \frac{\mathbf{p}^k(o,y)\mathbf{p}^{n-k}(y,x)}{\mathbf{p}^n(o,x)} \gamma(k,y)\,. \tag{4.1}$$

For $a = 1, \dots, 6$ we define $\mathcal{E}_{(\mathrm{a})}$ to be the event that conditions (1) to $(a-1)$ in Definitions 2.2 and 2.4 are satisfied, but condition (a) is not. Then we may write the disjoint union

$$\mathcal{A}(i)^c \cup \mathcal{B}(i,c_0)^c = \bigcup_{a=1}^{6} \mathcal{E}_{(\mathrm{a})} \cup (\mathcal{A}(i) \cap B(i,c_0)^c)\ .$$

Recall the constants n_5, n_7 introduced in (3.2) and Lemma 3.2.

Lemma 4.1. *There exist $C_4 > 0$ and $\delta_2 > 0$ such that*

$$\mathbf{E}\Big[R_{\mathrm{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{E}_{(1)}, \Phi(V_n) = (x,n)\Big] \le (1 + C_4\delta)\bar{\gamma}(\delta n;(x,n)) + (K-1)\bar{\gamma}(\delta n;(x,n))$$

whenever $0 < \delta < \delta_2$, $\delta n \ge \max\{n_5(C_3), n_7(C_3)\}$.

Proof. By the triangle inequality of effective resistance (1.2) we have

$$\begin{aligned}\mathbf{E}\Big[R_{\mathrm{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{E}_{(1)}, \Phi(V_n) = (x,n)\Big] \\ \le \sum_{i'=i}^{i+K-1} \mathbf{E}\Big[R_{\mathrm{eff}}(\Phi(X_{i'}) \leftrightarrow \Phi(X_{i'+1}))\,\Big|\,\mathcal{E}_{(1)}, \Phi(V_n) = (x,n)\Big]\,.\end{aligned} \tag{4.2}$$

Each of the terms $i' = i+1, \dots, i+K-1$ is bounded by $\bar{\gamma}(\delta n;(x,n))$ since for such i' conditioning on $\mathcal{E}_{(1)}$ leaves the distribution of the tree between $X_{i'}$ and $X_{i'+1}$ unaltered. Hence we get the $(K-1)\bar{\gamma}(\delta n;(x,n))$ term. So it remains to prove that

$$\mathbf{E}\Big[R_{\mathrm{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1}))\mathbf{1}_{\mathcal{E}_{(1)}}\ \mid\ \Phi(V_n) = (x,n)\Big] \le (1 + O(\delta))\bar{\gamma}(\delta n;(x,n))\mathbf{P}(\mathcal{E}_{(1)} | \Phi(V_n) = (x,n))\,.$$

If $\mathcal{E}_{(1)}$ occurs, then precisely one of the following three disjoint events must happen:

(i) There are no levels in $[i\delta n, (i+1)\delta n)$ that reach height $(i+2)\delta n$,
(ii) There is more than one such level,
(iii) There is a unique such level ℓ_1 but $\ell_1 \notin [(i+1/4)\delta n, (i+3/4)\delta n]$.

We handle each of these separately. If (i) occurs, then the trees emanating from each level are conditioned not to reach level $(i+2)\delta n$. Hence,

$$\begin{aligned}\mathbf{E}\Big[R_{\mathrm{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1}))\mathbf{1}_{\text{(i)}} \;\mid\; \Phi(V_n)=(x,n)\Big] \\ \le \bar{\gamma}(\delta n;(x,n))\mathbf{P}(\text{(i)} \,|\, \Phi(V_n)=(x,n))\,,\end{aligned}$$

since in the definition of $\bar{\gamma}(\delta n;(x,n))$ we take a supremum over $m \ge 2\delta n$.

In handling the event (ii), the following notation will be convenient. We write $R_{\mathrm{eff}}(\Phi(X_i) \overset{\mathcal{G}}{\leftrightarrow} \Phi(X_{i+1}))$ for the effective resistance evaluated in a given graph $\mathcal{G}$. If (ii) occurs, then let $j_1,\ldots,j_k$ be the set of levels in $[i\delta n,(i+1)\delta n)$ such that $k\ge 2$ and $\mathcal{T}_{n,m}(j_s)$ reaches level $(i+2)\delta n$ but not level m for all $s=1,\ldots,k$ and denote by $\mathcal{F}(j_1,\ldots,j_k)$ this event. Let $\mathcal{T}_{n,\infty}$ be defined as $\mathcal{T}_{n,m}$ only without the conditioning on the side branches. We have

$$\begin{aligned}&\mathbf{E}\Big[R_{\mathrm{eff}}\Big(\Phi(X_i) \overset{\mathcal{T}_{n,m}}{\leftrightarrow} \Phi(X_{i+1})\Big) \,\Big|\, \text{(ii)},\, \Phi(V_n)=(x,n)\Big] \\ &\quad= \sum_{\substack{k\ge 2\\(j_1,\ldots,j_k)}} \mathbf{E}\Big[R_{\mathrm{eff}}\Big(\Phi(X_i) \overset{\mathcal{T}_{n,\infty}}{\leftrightarrow} \Phi(X_{i+1})\Big) \,\Big|\, \mathcal{F}(j_1,\ldots,j_k),\, \Phi(V_n)=(x,n)\Big] \\ &\qquad\qquad \times \mathbf{P}(\mathcal{F}(j_1,\ldots,j_k) \,|\, \text{(ii)})\,,\end{aligned}$$

since the events in question require that all side branches emanating from $V_{i\delta n}$ to $V_{(i+K)\delta n}$ do not reach level m. During the rest of the proof of (ii) we work where $\mathcal{T}_{n,\infty}$ is the background measure.

Write $\mathcal{F}'(j_1,\ldots j_k)$ for the same event as $\mathcal{F}(j_1,\ldots,j_k)$ except that the trees $\mathcal{T}_{n,\infty}(j_s)$ are now only required to reach level $(i+2)\delta n$ (and may perhaps reach level m as well). Since $\mathcal{F} \subseteq \mathcal{F}'$ we have

$$\begin{aligned}&\mathbf{E}\Big[R_{\mathrm{eff}}(\Phi(X_i)\leftrightarrow\Phi(X_{i+1}))\mathbf{1}_{\mathcal{F}(j_1,\ldots,j_k)} \,\Big|\, \Phi(V_n)=(x,n)\Big] \\ &\qquad \le \mathbf{E}\Big[R_{\mathrm{eff}}(\Phi(X_i)\leftrightarrow\Phi(X_{i+1}))\mathbf{1}_{\mathcal{F}'(j_1,\ldots,j_k)} \,\Big|\, \Phi(V_n)=(x,n)\Big]\,.\end{aligned} \tag{4.3}$$

Since $\mathcal{F}'$ is an increasing event and $R_{\mathrm{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1}))$ is a decreasing random variable, the FKG inequality [5, 4] implies that the right hand side of (4.3) is at most

$$\begin{aligned}&\mathbf{E}\Big[R_{\mathrm{eff}}(\Phi(X_i) \overset{\mathcal{T}_{n,\infty}}{\leftrightarrow} \Phi(X_{i+1})) \,\Big|\, \Phi(V_n)=(x,n)\Big]\mathbf{P}(\mathcal{F}'(j_1,\ldots,j_k)) \\ &\qquad \le \mathbf{P}(\mathcal{F}'(j_1,\ldots,j_k))\,\bar{\gamma}(\delta n;(x,n))\,,\end{aligned}$$

where in the last step we are using that $\mathcal{T}_{n,\infty}$ is the weak limit as $m\to\infty$ of $\mathcal{T}_{n,m}$.

We need to bound the ratio between the probability of $\mathcal{F}$ and $\mathcal{F}'$. Write $\mathcal{N}$ for the total number of progeny at level $(i+2)\delta n$ of $\mathcal{T}_{n,\infty}(j_1),\ldots,\mathcal{T}_{n,\infty}(j_k)$.

Then,

$$\begin{aligned}\mathbf{P}(\mathcal{F}) &\geq \mathbf{P}(\mathcal{F}')\mathbf{E}\big[(1-\theta(m-(i+2)\delta n))^{\mathcal{N}} \,\big|\, \mathcal{F}'\big] \\ &\geq \mathbf{P}(\mathcal{F}')\mathbf{E}\Big[\left(1-6(\sigma^2 n)^{-1}\right)^{\mathcal{N}} \,\Big|\, \mathcal{F}'\Big],\end{aligned}$$

where the last inequality is by $m-(i+2)\delta n \geq (1-2\delta)n \geq n/2 \geq n_5$ and our estimate on θ in (3.2). Note that $\mathcal{N} = \mathcal{N}^{(1)}+\ldots+\mathcal{N}^{(k)}$ where $\mathcal{N}^{(1)},\ldots,\mathcal{N}^{(k)}$ are independent and $\mathcal{N}^{(s)}$ has the distribution of $\widetilde{\mathcal{N}}_{(i+2)\delta n-j_s}$, $s=1,\ldots,k$. Hence,

$$\begin{aligned}\mathbf{E}\Big[\left(1-6(\sigma^2 n)^{-1}\right)^{\mathcal{N}} \Big|\mathcal{F}'\Big] &\geq \prod_{s=1}^{k}\mathbf{E}\Big[\left(1-6(\sigma^2 n)^{-1}\right)^{\mathcal{N}^{(s)}} \Big|\mathcal{N}^{(s)}>0\Big] \\ &\geq \prod_{s=1}^{k}\mathbf{E}\big[(1-\mathcal{N}^{(s)}6(\sigma^2 n)^{-1}) \mid \mathcal{N}^{(s)}>0\big] \\ &\geq (1-O(\delta))^k,\end{aligned}$$

since $\mathbf{E}[\mathcal{N}^{(s)}|\mathcal{N}^{(s)}>0] = \mathbf{E}\widetilde{\mathcal{N}}_{2\delta n-j_s}\mathbf{P}(\widetilde{\mathcal{N}}_{2\delta n-j_s}>0)^{-1} = O(\sigma^2\delta n)$ when $\delta n \geq n_7$, by (3.5). Hence,

$$\mathbf{P}(\mathcal{F}'(j_1,\ldots,j_k)) \leq (1+O(\delta))^k\mathbf{P}(\mathcal{F}(j_1,\ldots,j_k)).$$

Therefore,

$$\begin{aligned}&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \overset{\mathcal{T}_{n,m}}{\leftrightarrow} \Phi(X_{i+1})) \mid \mathcal{F}(j_1,\ldots,j_k), \Phi(V_n)=(x,n)\Big] \\ &\qquad \leq (1+O(\delta))^k\bar{\gamma}(\delta n;(x,n)).\end{aligned}$$

In the tree $\mathcal{T}_{n,\infty}$, and hence in the tree $\mathcal{T}_{n,m}$, the number of vertices V_k on the backbone that reach $(i+2)\delta n$ is stochastically bounded above by a Binomial random variable with parameters δn and $p=\frac{C}{\delta n}$, by (3.5). Hence, the probability that there are precisely k such vertices is at most e^{-ck} for some $c>0$. We get that as long as $\delta>0$ is small enough (as a function of c) we have

$$\begin{aligned}&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \overset{\mathcal{T}_{n,m}}{\leftrightarrow} \Phi(X_{i+1}))\mathbf{1}_{\text{(ii)}} \mid \Phi(V_n)=(x,n)\Big] \\ &\qquad \leq (1+O(\delta))\bar{\gamma}(\delta n;(x,n))\mathbf{P}(\text{(ii)} \,|\, \Phi(V_n)=(x,n)),\end{aligned}$$

concluding the analysis of (ii).

If (iii) occurs, then there is a unique ℓ_1 which reaches level $(i+2)\delta n$ but not m and all other levels do not reach level $(i+2)\delta n$. A similar analysis as in (ii) with $k=1$ using the FKG inequality gives that

$$\begin{aligned}&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1}))\mathbf{1}_{\text{(iii)}} \mid \Phi(V_n)=(x,n)\Big] \\ &\qquad \leq (1+O(\delta))\bar{\gamma}(\delta n;(x,n))\mathbf{P}(\text{(iii)} \,|\, \Phi(V_n)=(x,n)).\end{aligned}$$

□

Lemma 4.2.

$$\begin{aligned}\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{E}_{(2)},\,\ell_1,\,\Phi(V_n)=(x,n)\Big] \\ \le \bar{\gamma}(\ell_1 - i\delta n;(x,n)) + 1 + \bar{\gamma}((i+1)\delta n - \ell_1 - 1) \\ + (K-1)\bar{\gamma}(\delta n;(x,n))\,.\end{aligned}$$

Proof. As in the previous lemma, we use the triangle inequality as in (4.2) with $\mathcal{E}_{(1)}$ now replaced by $\mathcal{E}_{(2)}$. By the same reasoning as below (4.2), the sum of the terms containing $R_{\text{eff}}(\Phi(X_{i'}) \leftrightarrow \Phi(X_{i'+1}))$ for $i' = i+1,\dots,i+K-1$ are bounded $(K-1)\bar{\gamma}(\delta n;(x,n))$. The rest of the lemma is much easier than the previous one, since on the event that Definition 2.2(1) is satisfied, the backbone $V_{i\delta n},\dots,V_{\ell_1}$ together with its side branches (not counting the side branch of V_{ℓ_1}) is distributed as $\mathcal{T}_{\ell_1 - i\delta n, 2\delta n}$, and the backbone $V_{\ell_1+1},\dots V_{(i+1)\delta n}$ together with its side branches (again, not counting the side branch of $V_{(i+1)\delta n}$) is distributed as $\mathcal{T}_{(i+1)\delta n - \ell_1 - 1, 2\delta n - \ell_1 - 1}$. Hence we get

$$\begin{aligned}\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1}))\,\Big|\,\mathcal{E}_{(2)},\,\ell_1,\,\Phi(V_n)=(x,n)\Big] \\ \le \bar{\gamma}(\ell_1 - i\delta n;(x,n)) + 1 + \bar{\gamma}((i+1)\delta n - \ell_1 - 1)\,,\end{aligned}$$

as required. □

Lemma 4.3. *We have*

$$\begin{aligned}\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{E}_{(3)},\,\ell_1,\,\Phi(V_n)=(x,n)\Big] \\ \le \bar{\gamma}(\ell_1 - i\delta n;(x,n)) + 1 + \bar{\gamma}((i+1)\delta n - \ell_1 - 1) \\ + (K-2)\bar{\gamma}(\delta n;(x,n)) + (1+C_4\delta)\bar{\gamma}(\delta n;(x,n))\end{aligned}$$

whenever $0 < \delta < \delta_2$, $\delta n \ge \max\{n_5(C_3), n_7(C_3)\}$.

Proof. We again start with the triangle inequality as in (4.2), with $\mathcal{E}_{(1)}$ now replaced by $\mathcal{E}_{(3)}$. An argument almost identical to that of Lemma 4.1, yields that the term involving $R_{\text{eff}}(\Phi(X_{i+K-1}) \leftrightarrow \Phi(X_{i+K}))$ is bounded by $(1+C_4\delta)\bar{\gamma}(\delta n;(x,n))$. The rest of the terms are bounded as in Lemma 4.2. □

Lemma 4.4.

$$\begin{aligned}\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{E}_{(4)},\,\ell_1,\,\ell_2,\,\Phi(V_n)=(x,n)\Big] \\ \le \bar{\gamma}(\ell_1 - i\delta n;(x,n)) + 1 + \bar{\gamma}((i+1)\delta n - \ell_1 - 1) \\ + (K-2)\bar{\gamma}(\delta n;(x,n)) + \bar{\gamma}(\ell_2 - ((i+K-1)\delta n;(x,n)) + 1 \\ + \bar{\gamma}((i+K)\delta n - \ell_2 - 1;(x,n))\,.\end{aligned}$$

Proof. Similarly to the proof of Lemma 4.2, we bound the expected resistance using subgraphs that conditioned on Definition 2.2(1),(2),(3) holding (and conditioned on the values of ℓ_1, ℓ_2) are independent of whether (4) holds or not. □

5. Analysis of spatially bad blocks

In this section we analyze what happens when condition (5) or (6) in Definition 2.4 fails, that is, some spatial displacement is "not typical", and also what happens when $\mathcal{B}(i, c_0)$ fails. Let us introduce some notation. We write $\mathcal{G}_{\rm tree}$ for the event

$$\mathcal{G}_{\rm tree} = \{(1)\text{–}(4), \ell_1, \ell_2, \Phi(V_n) = (x, n)\}\,.$$

We define a set of times $i\delta n = T_0 < T_1 < \cdots < T_{K+4} = (i+K)\delta n$, time differences $t_1, t_2, \dots, t_{K+4}$ and spatial locations $z_0, \dots, z_{K+4} \in \mathbb{Z}^d$ by

$$\begin{array}{lll}
z_0 = x_i & & T_0 = i\delta n \\
z_1 = v_{\ell_1} & t_1 = \ell_1 - i\delta n & T_1 = \ell_1 \\
z_2 = v_{\ell_1+1} & t_2 = 1 & T_2 = \ell_1 + 1 \\
z_3 = x_{i+1} & t_3 = (i+1)\delta n - \ell_1 - 1 & T_3 = (i+1)\delta n \\
z_4 = x_{i+2} & t_4 = \delta n & T_4 = (i+2)\delta n \\
z_5 = x_{i+3} & t_5 = \delta n & T_5 = (i+3)\delta n \\
\vdots & \vdots & \vdots \\
z_{K+1} = x_{i+K-1} & t_{K+1} = \delta n & T_{K+1} = (i+K-1)\delta n \\
z_{K+2} = v_{\ell_2} & t_{K+2} = \ell_2 - (i+K-1)\delta n & T_{K+2} = \ell_2 \\
z_{K+3} = v_{\ell_2+1} & t_{K+3} = 1 & T_{K+3} = \ell_2 + 1 \\
z_{K+4} = x_{i+K} & t_{K+4} = K\delta n - \ell_2 - 1 & T_{K+4} = (i+K)\delta n
\end{array}$$

Observe that conditional on $\mathcal{G}_{\rm tree}$, the times T_s and time differences t_s are non-random but the spatial locations z_s are random. Furthermore, we define for any $s = 1, \dots, K+4$

$$\mathbf{q}_s(z) = \sum_{\substack{\|y_r\| \le \sqrt{t_r} \\ r=1,\dots,s-1 \\ y_1+\cdots+y_{s-1}=z}} \prod_{r=1}^{s-1} \mathbf{p}^{t_r}(0, y_r)\,. \tag{5.1}$$

Finally, for any $s = 1, \dots, K+4$ we define the event $\mathcal{E}^s_{(5)}$ by

$$\mathcal{E}^s_{(5)} = \bigcap_{r=1}^{s-1} \Big\{ \|z_r - z_{r-1}\| \le \sqrt{t_r} \Big\} \bigcap \Big\{ \|z_s - z_{s-1}\| > \sqrt{t_s} \Big\}\,.$$

Note that

$$\mathbf{P}(\mathcal{E}^s_{(5)} \,|\, \mathcal{G}_{\rm tree}) = \sum_{z,y: \|y\| > \sqrt{t_s}} \mathbf{q}_s(z) \frac{\mathbf{p}^{t_s}(z, z+y)\mathbf{p}^{n-(T_s-T_0)}(z+y, x)}{\mathbf{p}^n(o, x)}\,. \tag{5.2}$$

Lemma 5.1. *For any $s = 1, \dots, K+4$ and $s' = 1, \dots, K+4$ the quantity*

$$\mathcal{R}_{s',s} = \mathbf{E}\Big[R_{\rm eff}((z_{s'-1}, T_{s'-1}) \leftrightarrow (z_{s'}, T_{s'})) \mathbf{1}_{\mathcal{E}^s_{(5)}} \,\Big|\, \mathcal{G}_{\rm tree} \Big]$$

satisfies:

$$\mathcal{R}_{s',s} \leq \mathbf{P}(\mathcal{E}^s_{(5)} \,|\, \mathcal{G}_{\text{tree}}) \,, \quad \textit{when } s' = 2,\, K+3,$$

$$\mathcal{R}_{s',s} \leq \mathbf{P}(\mathcal{E}^s_{(5)} \,|\, \mathcal{G}_{\text{tree}})\gamma(t_{s'}) \,, \quad \textit{when } s' < s,\, s' \neq 2,\, K+3,$$

$$\mathcal{R}_{s,s'} \leq \sum_z \mathbf{q}_s(z) \sum_{y:\|y\|>\sqrt{t_s}} \frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s+T_0}(y,x-z)}{\mathbf{p}^n(o,x)}\gamma(t_s,y) \,,$$

when $s' = s,\, s' \neq 2,\, K+3,$

$$\mathcal{R}_{s',s} \leq \sum_{\substack{z,y_s,y:\\ \|y_s\|>\sqrt{t_s}}} \mathbf{q}_s(z) \frac{\mathbf{p}^{t_s}(o,y_s)\mathbf{p}^{t_{s'}}(o,y)\mathbf{p}^{n-t_{s'}-T_s}(y,x-z-y_s)}{\mathbf{p}^n(o,x)}\gamma(t_{s'},y) \,,$$

when $s' > s,\, s' \neq 2,\, K+3.$

Proof. The case $s' = 2,\, K+3$ is trivial, so assume $s' \neq 2,\, K+3$. The case $s' < s$ is easy. Condition on $\mathcal{E}^s_{(5)}$ and on the spatial locations $z_0, z_1, \dots, z_s$ such that $\mathcal{E}^s_{(5)}$ holds. Since $\|z_{s'} - z_{s'-1}\| \leq \sqrt{t_{s'}}$ we may bound the expected resistance between the corresponding points by $\gamma(t_{s'})$.

In order to handle the case $s' = s$, we condition on $z_0, z_1, \dots, z_s$ such that $\mathcal{E}^s_{(5)}$ holds. With this conditioning the required expected resistance is bounded above by $\gamma(t_s, z_s - z_{s-1})$. So the required expectation is bounded above by

$$\sum_{\substack{z_0,z_1,\dots,z_s:\\ \|z_r-z_{r-1}\|\leq\sqrt{t_r}\\ r=1,\dots,s-1\\ \|z_s-z_{s-1}\|>\sqrt{t_s}}} \frac{\mathbf{p}^{T_0}(o,z_0)}{\mathbf{p}^n(o,x)} \Big[\prod_{r=1}^{s} \mathbf{p}^{t_r}(z_{r-1},z_r)\Big] \mathbf{p}^{n-T_s}(z_s,x)\gamma(t_s,z_s-z_{s-1}) \,.$$

By changing variables $y_1 = z_1 - z_0, y_2 = z_2 - z_1, \dots, y_{s-1} = z_{s-1} - z_{s-2}$ and $y = z_s - z_{s-1}$ and $z = y_1 + \cdots + y_{s-1}$ this equals

$$\sideset{}{'}\sum_{z_0,y_1,\dots,y_{s-1},y} \frac{\mathbf{p}^{T_0}(o,z_0)}{\mathbf{p}^n(o,x)} \Big[\prod_{r=1}^{s-1} \mathbf{p}^{t_r}(o,y_r)\Big] \mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s}(z_0+z+y,x)\gamma(t_s,y) \,,$$

where $\sum'$ indicates the restriction $\|y_1\| \leq \sqrt{t_1}, \dots, \|y_{s-1}\| \leq \sqrt{t_s}, \|y\| > \sqrt{t_s}$. By summing over z_0 this simplifies to

$$\sum_z \mathbf{q}_s(z) \sum_{y:\|y\|>\sqrt{t_s}} \frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s+T_0}(y,x-z)}{\mathbf{p}^n(o,x)}\gamma(t_s,y)$$

as required.

The case $s' > s$ is done similarly. The required expectation is bounded above by

$$\sum_{\substack{z_0,z_1,\dots,z_s:\\ \|z_r-z_{r-1}\|\le\sqrt{t_r}\\ r=1,\dots,s-1\\ \|z_s-z_{s-1}\|>\sqrt{t_s}}} \sum_{z_{s'-1},y\in\mathbb{Z}^d} \frac{\mathbf{p}^{T_0}(o,z_0)}{\mathbf{p}^n(o,x)}\Big[\prod_{r=1}^{s}\mathbf{p}^{t_r}(z_{r-1},z_r)\Big]\mathbf{p}^{T_{s'-1}-T_s}(z_s,z_{s'-1})$$
$$\times\mathbf{p}^{t_{s'}}(z_{s'-1},z_{s'-1}+y)\mathbf{p}^{n-T_{s'}}(z_{s'-1}+y,x)\gamma(t_{s'},y)\,.$$

Summing over $z_0, z_{s'-1}$ and recalling (5.1) simplifies this to

$$\sum_{\substack{z,y_s,y:\\ \|y_s\|>\sqrt{t_s}}} \mathbf{q}_s(z)\frac{\mathbf{p}^{t_s}(o,y_s)\mathbf{p}^{t_{s'}}(o,y)\mathbf{p}^{n-t_{s'}-T_s}(y,x-z-y_s)}{\mathbf{p}^n(o,x)}\gamma(t_{s'},y)$$

□

Next, to handle part (6) of Definition 2.4 recall that we defined

$$\mathcal{E}_{(6)} = \bigcap_{s=1}^{K+4}\big\{\|z_s-z_{s-1}\|\le\sqrt{t_s}\big\}\bigcap\big\{\text{one of the conditions in (6) fails}\big\}\,.$$

We also define $\mathcal{E}_{(7)} = \mathcal{A}(i)\cap\mathcal{B}(i,c_0)^c$.

Lemma 5.2.

$$\begin{aligned}&\mathbf{E}\Big[R_{\mathrm{eff}}(\Phi(X_i)\leftrightarrow\Phi(X_{i+K}))\,\Big|\,\mathbf{1}_{\mathcal{E}_{(6)}\cup\mathcal{E}_{(7)}},\mathcal{G}_{\mathrm{tree}}\Big]\\ &\quad\le (K-2)\gamma(\delta n)+\gamma(\ell_1-i\delta n)+1+\gamma((i+1)\delta n-\ell_1-1)\\ &\qquad+\gamma(\ell_2-(i+K-1)\delta n)+1+\gamma((i+K)\delta n-\ell_2-1)\,.\end{aligned}$$

Proof. Condition on $\mathcal{G}_{\mathrm{tree}}$ and $\mathcal{E}_{(6)}\cup\mathcal{E}_{(7)}$. We have that $\|z_s-z_{s-1}\|\le\sqrt{t_s}$ for all $s=1,\dots,K+4$. Hence, under this conditioning, we may bound the expected resistance between (z_{s-1},T_{s-1}) and (z_s,T_s) by $\gamma(t_s)$, concluding the proof. □

We close this section with a bound on the expected resistance in the "final stretch" between $X_{i^{\mathrm{last}}}$ and V_n, where $i^{\mathrm{last}} = KN$ with $n = NK\delta n + K'\delta n + \tilde{n}$, $0\le K' < K$, $\delta n\le\tilde{n}<2\delta n$.

Lemma 5.3.

$$\begin{aligned}&\mathbf{E}\Big[R_{\mathrm{eff}}(\Phi(X_{i^{\mathrm{last}}})\leftrightarrow\Phi(V_n))\,\Big|\,\Phi(V_n)=(x,n)\Big]\\ &\quad\le K'\bar{\gamma}(\delta n;(x,n))+\bar{\gamma}(\tilde{n};(x,n))\,.\end{aligned}$$

Proof. This follows from the triangle inequality for resistance. □

6. Analysis of good blocks

In this section we will estimate expectations of resistances given the event

$$\mathcal{G}_{\text{good}} = \left\{\Phi(V_n) = (x,n), \mathcal{A}(i), \mathcal{B}(i,c_0), \ell_1, \ell_2\right\}.$$

Lemma 6.1. *Conditional on $\mathcal{G}_{\text{good}}$, we have*

(i) $\mathbf{E}\big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(V_{\ell_1})) \mid \mathcal{G}_{\text{good}}\big] \le \gamma(\ell_1 - i\delta n)$.
(ii) $\mathbf{E}\big[R_{\text{eff}}(\Phi(V_{\ell_1}) \leftrightarrow \Phi(X_{i+1})) \mid \mathcal{G}_{\text{good}}\big] \le \gamma((i+1)\delta n - \ell_1 - 1) + 1$.
(iii) *For all $i+1 \le j \le i+K-2$ we have*

$$\mathbf{E}\big[R_{\text{eff}}(\Phi(X_j) \leftrightarrow \Phi(X_{j+1})) \mid \mathcal{G}_{\text{good}}\big] \le \gamma(\delta n).$$

(iv) $\mathbf{E}\big[R_{\text{eff}}(\Phi(X_{i+K-1}) \leftrightarrow \Phi(V_{\ell_2})) \mid \mathcal{G}_{\text{good}}\big] \le \gamma(\ell_2 - (i+K-1)\delta n)$.
(v) $\mathbf{E}\big[R_{\text{eff}}(\Phi(V_{\ell_2}) \leftrightarrow \Phi(X_{i+K})) \mid \mathcal{G}_{\text{good}}\big] \le \gamma((i+K)\delta n - \ell_2 - 1) + 1$.
(vi) $\mathbf{E}\big[R_{\text{eff}}(\Phi(V_{\ell_2}) \leftrightarrow \Phi(X'_{i+K})) \mid \mathcal{G}_{\text{good}}\big] \le \gamma((i+K)\delta n - \ell_2 - 1) + 1$.
(vii) $\mathbf{E}\big[R_{\text{eff}}(\Phi(V_{\ell_1}) \leftrightarrow \Phi(Y_{i+1})) \mid \mathcal{G}_{\text{good}}\big] \le \gamma((i+1)\delta n - \ell_1 - 1) + 1$.
(viii) *For all $i+1 \le j \le i+K-1$ we have*

$$\mathbf{E}\big[R_{\text{eff}}(\Phi(Y_j) \leftrightarrow \Phi(Y_{j+1})) \mid \mathcal{G}_{\text{good}}\big] \le \gamma(\delta n).$$

Proof. The proof of (i), (iii), (iv) and (viii) is immediate by Definition 2.4. The other estimates follow almost as quickly by Definition 2.4 and triangle inequality for resistance. □

Recall the constant n_1 from Lemma 1.4(i) and the constant n_9 from Theorem 3.8.

Lemma 6.2. *Assume $d=5$. There exists $C_5 < \infty$ such that we have*

$$\mathbf{E}\big[R_{\text{eff}}(\Phi(X'_{i+K}) \leftrightarrow \Phi(Y_{i+K}) \mid \mathcal{G}_{\text{good}}\big] \le C_5 \max_{1 \le \ell \le \delta n} \gamma(\ell)$$

whenever $\delta n \ge \max\{n_1(\mathbf{p}^1), n_9(\sigma^2, C_3, \mathbf{p}^1)\}$.

For convenience we will prove Lemma 6.2 under the assumption that there exists an M such that the progeny distribution is bounded by M with probability 1. Then by taking $M \to \infty$ and keeping n fixed we obtain Lemma 6.2 in our usual generality. This is possible, since C_5 does not depend on M, and the restriction on δn only depends on σ^2, C_3, so it is sufficient to approximate $\{p(k)\}$ by some $\{p_M(k)\}$ in such a way that

$$1 = \sum_{k=0}^{M} p_M(k) = \sum_{k=0}^{M} k p_M(k),$$
$$\sigma^2 = \lim_{M\to\infty} \sum_{k\ge 1} k(k-1) p_M(k),$$
$$C_3 \ge \sup_{M \ge 1} \sum_{k \ge 1} k^3 p_M(k).$$

Therefore in the rest of this section we assume the bound M.

Given any n and m such that $m \geq 2n$ we regard the random tree $\mathcal{T}_{n,m}$ as a subtree of an infinite M-ary tree T_M with root ρ as follows: the root of $\mathcal{T}_{n,m}$ is mapped to ρ and if W is a vertex of $\mathcal{T}_{n,m}$ with k children we map the k edges randomly amongst the $\binom{M}{k}$ possible choices in T_M. Denote by $\mathcal{V}_n \in T_M$ the random vertex where the last backbone vertex of $\mathcal{T}_{n,m}$ was mapped to. The triple $(\mathcal{T}_{n,m}, \rho, \mathcal{V}_n)$ is a doubly rooted tree. Define

$$q(k) := \binom{M}{k}^{-1} p(k) .$$

Lemma 6.3. *For a fixed triple (t, ρ, V) where $t \subset T_M$ is a tree and $V \in T_M$ at height n such that t does not reach level m and V has no children in t, we have*

$$\mathbf{P}((\mathcal{T}_{n,m}, \rho, \mathcal{V}_n) = (t, \rho, V)) \propto \prod_{\substack{W \in t \\ W \neq V}} q(\deg_t^+(W)) , \tag{6.1}$$

where $\deg_t^+(W)$ is the number of children of W in t.

Proof. Let $\rho = V_0, V_1, \dots, V_n = V$ be the unique path in t from ρ to V. The probability that $\mathcal{T}_{n,m} = t$ with this backbone equals

$$\frac{1}{Z} \prod_{i=0}^{n-1} \tilde{p}(\deg_t^+(V_i) - 1) \cdot \prod_{W \in t \setminus \{V_0, \dots, V_n\}} p(\deg_t^+(W)) ,$$

where $Z = \prod_{i=0}^{n-1} \theta(m - i)$. Hence, the probability that $(\mathcal{T}_{n,m}, \rho, \mathcal{V}_n) = (t, \rho, V)$ (as subtrees of T_M) equals

$$\frac{1}{Z} \prod_{i=0}^{n-1} \left[M \binom{M-1}{\deg_t^+(V_i) - 1} \right]^{-1} \tilde{p}(\deg_t^+(V_i) - 1)$$
$$\times \prod_{W \in t \setminus \{V_0, \dots, V_n\}} \left[\binom{M}{\deg_t^+(W)} \right]^{-1} p(\deg_t^+(W)) .$$

Manipulating with $\tilde{p}(k-1) = kp(k)$ finishes the proof. □

For the statement of the next lemma we fix

$$0 \leq k_1 \leq \delta n - 1 \qquad k_1 + 1 \leq h_u \leq \delta n$$

Given $V \in T_M$ at level δn and $U \in T_M$ at level h_u let $Z \in T_M$ be the highest common ancestor of V and U and let Z^+ be the unique child of Z leading towards U. Given a tree $t \subset T_M$ such that $V, U \in t$ and V does not have any children in t, we have a unique decomposition of t into edge disjoint trees $(t^A, \rho, Z), (t^B, Z^+, U), t^C$ and t^D, see Figure 4. The doubly rooted tree (t^A, ρ, Z) contains all the descendants of ρ that are not descendants of Z. The doubly rooted tree (t^B, Z^+, U) contains all the descendants of Z^+ that are not descendants of U. The tree t^C contains all the descendants of U and finally the tree t^D contains all other edges, namely, all the descendants of Z that are not descendants of Z^+ (in particular, the edge Z, Z^+ is in t^D).

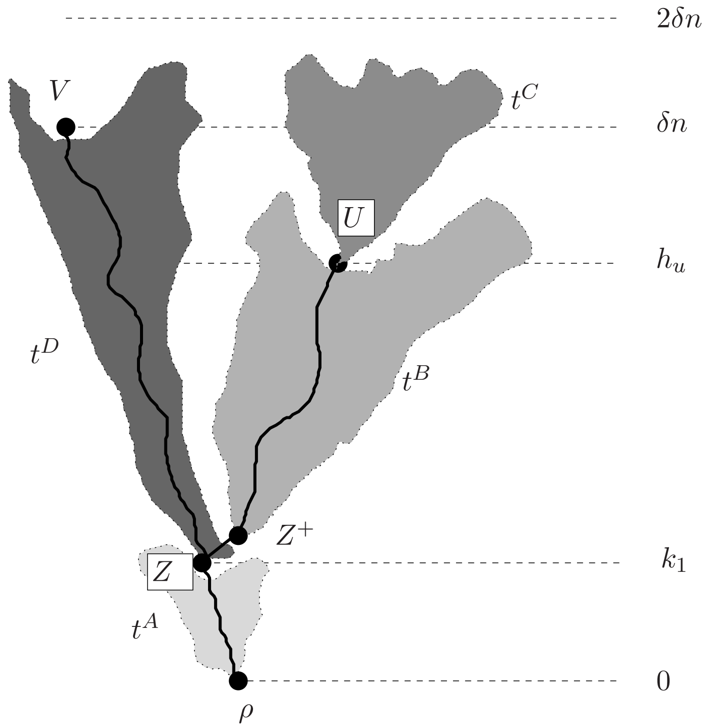

FIGURE 4. Illustration of the decomposition into edge-disjoint trees t^A, t^B, t^C, t^D appearing in Lemma 6.4 ($2\delta n$ and δn are not to scale).

For $W \in T_M$ let Θ_W denote the tree isomorphism that takes W to ρ and the descendants subtree of W onto T_M.

Lemma 6.4. *Let $V, U \in T_M$ be at heights δn and h_u, respectively and $(\mathcal{T}, \rho, \mathcal{V})$ be distributed as $(\mathcal{T}_{\delta n, 2\delta n}, \rho, \mathcal{V}_{\delta n})$. Conditionally on the event $\{\mathcal{V} = V, U \in \mathcal{T}\}$ we have that*

$$(\mathcal{T}^A, \rho, Z) \stackrel{\mathrm{d}}{=} (\mathcal{T}_{k_1, 2\delta n}, \rho, \mathcal{V}_{k_1}) \mid \mathcal{V}_{k_1} = Z \,,$$

and

$$\Theta_{Z^+}((\mathcal{T}^B, Z^+, U)) \stackrel{\mathrm{d}}{=} (\mathcal{T}_{h_u - k_1 - 1, 2\delta n - k_1 - 1}, \rho, \mathcal{V}_{h_u - k_1 - 1}) \mid \mathcal{V}_{h_u - k_1 - 1} = \Theta_{Z^+}(U) \,.$$

Proof. For any $t \subset T_M$ that contains V and U (and V has no children in t) by Lemma 6.3 we have

$$\mathbf{P}((\mathcal{T}, \rho, \mathcal{V}) = (t, \rho, V)) \propto \prod_{\substack{W \in t \\ W \neq V}} q(\deg_t^+(W)) \,.$$

We factorize the right hand side so it equals

$$\prod_{\substack{W \in t^A \\ W \neq Z}} q(\deg_t^+(W)) \cdot \prod_{\substack{W \in t^B \\ W \neq U}} q(\deg_t^+(W)) \cdot \prod_{W \in t^C} q(\deg_t^+(W)) \cdot \prod_{\substack{W \in t^D \\ W \neq V}} q(\deg_t^+(W)) \,.$$

By summing over all the possible values of t^B, t^C and t^D we get that

$$\mathbf{P}((\mathcal{T}^A, \rho, Z) = (t^A, \rho, Z)) \propto \prod_{\substack{W \in t^A \\ W \neq Z}} q(\deg_t^+(W)),$$

which gives the claim for $\mathcal{T}^A$ by Lemma 6.3. The same argument works similarly for $\mathcal{T}^B$ noting that under the shift Θ_{Z^+} the degrees do not change. $\square$

Proof of Lemma 6.2. All our expectations in the following proof are conditioned on the event $\mathcal{A}(i), \Phi(V_n) = (x, n)$.

Let $(\mathcal{T}_1, \rho, \mathcal{V}_1)$ and $(\mathcal{T}_2, \rho, \mathcal{V}_2)$ be two independent copies of $(\mathcal{T}_{\delta n, 2\delta n}, \rho, \mathcal{V}_{\delta n})$ randomly embedded into T_M as before. Conditionally on $\mathcal{A}(i)$ let Φ_1 and Φ_2 be independent random walk mappings of T_M such that $\Phi_1(\rho) = \Phi(\mathcal{X}'_{i+K})$ and $\Phi_2(\rho) = \Phi(\mathcal{Y}_{i+K})$, so that $\|\Phi_1(\rho) - \Phi_2(\rho)\| \leq \sqrt{\delta n}$. In this way, the required quantity $R_{\text{eff}}(\Phi(\mathcal{X}'_{i+K}) \leftrightarrow \Phi(\mathcal{Y}_{i+K}))$ (computed in the Φ-image of the union of the subtrees emanating from $\mathcal{X}'_{i+K}$ and $\mathcal{Y}_{i+K}$) is distributed as $R_{\text{eff}}(\Phi_1(\rho) \leftrightarrow \Phi_2(\rho))$ (computed in the graph $\Phi_1(\mathcal{T}_1) \cup \Phi_2(\mathcal{T}_2)$).

For notational convenience, and without loss of generality, we assume that $\Phi_1(\rho_1) = (o, 0)$, $\Phi_2(\rho_2) = (x, 0)$, with $\|x\| \leq \sqrt{\delta n}$. Recall the notation Z_1, Z_1^+, Z_2, Z_2^+ introduced after Definition 2.5. Definition 2.6 adapted to the current setting reads as follows:

Definition 6.1. *We say that the vertices $U_1, U_2 \in T_M$* intersect-well *if:*
1. $U_1 \in \mathcal{T}_1$ *and* $U_2 \in \mathcal{T}_2$*;*
2. $(5/6)\delta n \leq h(U_1) = h(U_2) \leq \delta n$*;*
3. $(1/2)\delta n \leq h(Z_1), h(Z_2) \leq (4/6)\delta n$*;*
4. $\mathcal{TS}(\rho_1, Z_1)$*,* $\mathcal{TS}(Z_1^+, U_1)$*,* $\mathcal{TS}(\rho_2, Z_2)$*,* $\mathcal{TS}(Z_2^+, U_2)$*;*
5. $\Phi_1(U_1) = \Phi_2(U_2)$*.*

Define $\tilde{\mathcal{I}}$ by

$$\tilde{\mathcal{I}} = \{(U_1, U_2) : U_1, U_2 \text{ intersect-well}\}. \tag{6.2}$$

Then it is clear that $|\tilde{\mathcal{I}}|$ has the same distribution as $|\mathcal{I}|$ introduced earlier. Recall that $\mathcal{B} = \mathcal{B}(i, c_0)$ is the event $\{|\mathcal{I}| \geq c_0 \frac{\sigma^4}{D^d} (\delta n)^{(6-d)/2}\}$. Conditional on $\mathcal{T}_1, \mathcal{T}_2, \Phi_1, \Phi_2$, and the event $\mathcal{B}$, draw a pair $(\mathcal{U}_1, \mathcal{U}_2)$ from the set $\tilde{\mathcal{I}}$, uniformly at random. This is possible, since on the event $\mathcal{B}$ we have $|\tilde{\mathcal{I}}| > 0$. Denote

$$\mathcal{R} = R_{\text{eff}}(\Phi_1(\rho_1) \leftrightarrow \Phi_2(\rho_2)).$$

Writing for short

$$\mathbf{1}_V = \mathbf{1}_{\mathcal{V}_1 = V_1, \mathcal{V}_2 = V_2},$$
$$\mathbf{1}_U = \mathbf{1}_{\mathcal{U}_1 = U_1, \mathcal{U}_2 = U_2},$$

we have

$$\mathbf{E}[\mathcal{R}\mathbf{1}_{\mathcal{B}}] = \sum_{V_1, V_2 \in T_M} \sum_{U_1, U_2 \in T_M} \mathbf{E}[\mathcal{R}\mathbf{1}_{\mathcal{B}} \mathbf{1}_V \mathbf{1}_U]. \tag{6.3}$$

Recall that V_1 and U_1 determine the vertices Z_1, Z_1^+, and V_2 and U_2 determine Z_2, Z_2^+. The first sum in (6.3) is over all pairs $V_1, V_2 \in T_M$ at height δn. The second sum is over all pairs $U_1, U_2 \in T_M$ such that

$$\begin{aligned}(5/6)\delta n \le h_u := h(U_1) = h(U_2) \le \delta n\,,\\ \delta n/2 \le k_1 := h(Z_1) \le (4/6)\delta n\,,\\ \delta n/2 \le k_2 := h(Z_2) \le (4/6)\delta n\,.\end{aligned}$$

Given $z_1, z_1^+, z_2, z_2^+, u \in \mathbb{Z}^d$, write $\mathbf{1}_\Phi$ for short for the indicator function of the intersection of the following six events:

$$\begin{aligned}&\Phi_1(Z_1) = (z_1, k_1) &&\Phi_2(Z_2) = (z_2, k_2)\\ &\Phi_1(Z_1^+) = (z_1^+, k_1+1) &&\Phi_2(Z_2^+) = (z_2^+, k_2+1)\\ &\Phi_1(U_1) = (u, h_u) &&\Phi_2(U_2) = (u, h_u)\end{aligned}$$

This allows us to rewrite (6.3) in the form:

$$\mathbf{E}[\mathcal{R}\mathbf{1}_\mathcal{B}] = \sum_{V_1,V_2}\sum_{U_1,U_2}\sideset{}{'}\sum_{\substack{u,z_1,z_1^+\\ z_2,z_2^+}} \mathbf{E}[\mathcal{R}\mathbf{1}_\mathcal{B}\mathbf{1}_V\mathbf{1}_U\mathbf{1}_\Phi]\,. \tag{6.4}$$

Here the prime on the summation over $z_1, z_1^+, z_2, z_2^+, u$ indicates that these vertices are restricted to choices that are compatible with the occurrence of $\mathcal{TS}(\rho_1, Z_1)$, $\mathcal{TS}(Z_1^+, U_1)$, $\mathcal{TS}(\rho_2, Z_2)$, $\mathcal{TS}(Z_2^+, U_2)$, that is, $\|z_1\| \le \sqrt{k_1}$, $\|u - z_1^+\| \le \sqrt{h_u - k_1 - 1}$, $\|z_2 - x\| \le \sqrt{k_2}$, $\|u - z_2^+\| \le \sqrt{h_u - k_2 - 1}$.

In the presence of the indicators on the right hand side of (6.4) we can also insert the indicator

$$\mathbf{1}_T = \mathbf{1}_{U_1 \in \mathcal{T}_1,\, U_2 \in \mathcal{T}_2}\,,$$

as this event already occurs. Hence the expectation on the right hand side of (6.4) equals

$$\mathbf{E}[\mathcal{R}\mathbf{1}_\mathcal{B}\mathbf{1}_V\mathbf{1}_U\mathbf{1}_T\mathbf{1}_\Phi]\,. \tag{6.5}$$

Observe that we have

$$\mathbf{E}[\mathbf{1}_U \,|\, \mathcal{B},\, (\mathcal{T}_1, \mathcal{V}_1), (\mathcal{T}_2, \mathcal{V}_2), \Phi_1, \Phi_2] = \frac{1}{|\tilde{\mathcal{I}}|} \le \frac{D^d}{c_0\sigma^4(\delta n)^{(6-d)/2}}\,, \tag{6.6}$$

where the last inequality is by definition of $\mathcal{B}$, and that $\mathcal{R}$ and the other indicators in (6.5) are measurable with respect to the conditioning in (6.6). Hence

$$\begin{aligned}\mathbf{E}[\mathcal{R}\mathbf{1}_\mathcal{B}\mathbf{1}_V\mathbf{1}_U\mathbf{1}_T\mathbf{1}_\Phi] &\le \frac{D^d}{c_0\sigma^4(\delta n)^{(6-d)/2}}\mathbf{E}[\mathcal{R}\mathbf{1}_\mathcal{B}\mathbf{1}_V\mathbf{1}_T\mathbf{1}_\Phi]\\ &\le \frac{D^d}{c_0\sigma^4(\delta n)^{(6-d)/2}}\mathbf{E}[\mathcal{R}\mathbf{1}_V\mathbf{1}_T\mathbf{1}_\Phi]\,.\end{aligned} \tag{6.7}$$

In order to bound $\mathcal{R}$ from above, we define

$$\mathcal{R}_1 = R_{\mathrm{eff}}(\Phi_1(\rho_1) \leftrightarrow \Phi_1(U_1)) \qquad \text{and} \qquad \mathcal{R}_2 = R_{\mathrm{eff}}(\Phi_2(\rho_2) \leftrightarrow \Phi_2(U_2))\,,$$

and by the triangle inequality for effective resistance (1.2) we have

$$\mathcal{R} \le \mathcal{R}_1 + \mathcal{R}_2\,,$$

on the event $\mathbf{1}_T \mathbf{1}_\Phi$. Inserting this into (6.5) yields

$$\mathbf{E}[\mathcal{R}\mathbf{1}_{\mathcal{B}}] \le \frac{D^d}{c_0\sigma^4(\delta n)^{(6-d)/2}} \sum_{\substack{V_1,V_2,\\ U_1,U_2}} {\sum_{\substack{u,z_1,z_1^+\\ z_2,z_2^+}}}' \Big(\mathbf{E}[\mathcal{R}_1\mathbf{1}_V\mathbf{1}_T\mathbf{1}_\Phi] + \mathbf{E}[\mathcal{R}_2\mathbf{1}_V\mathbf{1}_T\mathbf{1}_\Phi]\Big)\,.$$

We only analyze the term containing $\mathcal{R}_1$, since the arguments for handling $\mathcal{R}_2$ are identical. We bound $\mathcal{R}_1$ from above by,

$$\mathcal{R}_1 \le R_{\mathrm{eff}}(\Phi_1(\rho_1) \leftrightarrow \Phi_1(Z_1)) + 1 + R_{\mathrm{eff}}(\Phi_1(Z_1^+) \leftrightarrow \Phi_1(U_1))\,.$$

Due to Lemma 6.4, conditioned on the events in the indicators $\mathbf{1}_V\mathbf{1}_T$, the distribution of $\mathcal{T}_1^A$ is the same as that of $\mathcal{T}_{k_1,2\delta n}$, and the distribution of $\mathcal{T}_1^B$ is the same as the distribution of $\mathcal{T}_{h_u-k_1-1,2\delta n-k_1-1}$. Due to the presence of the indicator $\mathbf{1}_\Phi$, that fixes the spatial locations of $\Phi_1(Z_1), \Phi_1(Z_1^+), \Phi_1(U_1)$ (respectively) to be z_1, z_1^+, u (respectively), we have

$$\begin{aligned}\mathbf{E}[R_{\mathrm{eff}}(\Phi_1(\rho_1) \leftrightarrow \Phi_1(Z_1))\mathbf{1}_V\mathbf{1}_T\mathbf{1}_\Phi] &= \gamma(k_1, z_1)\mathbf{E}[\mathbf{1}_V\mathbf{1}_T\mathbf{1}_\Phi] \\ &\le \gamma(k_1)\mathbf{E}[\mathbf{1}_V\mathbf{1}_T\mathbf{1}_\Phi]\,,\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}[R_{\mathrm{eff}}(\Phi_1(Z_1^+) \leftrightarrow \Phi_1(U_1))\mathbf{1}_V\mathbf{1}_T\mathbf{1}_\Phi] &= \gamma(h_u - k_1 - 1, u - z_1^+)\mathbf{E}[\mathbf{1}_V\mathbf{1}_T\mathbf{1}_\Phi] \\ &\le \gamma(h_u - k_1 - 1)\mathbf{E}[\mathbf{1}_V\mathbf{1}_T\mathbf{1}_\Phi]\,.\end{aligned}$$

Together with analogous bounds for $\mathcal{R}_2$, this yields

$$\begin{aligned}\mathbf{E}[\mathcal{R}\mathbf{1}_{\mathcal{B}}] &\le \frac{D^d}{c_0\sigma^4(\delta n)^{(6-d)/2}} \sum_{\substack{V_1,V_2,\\ U_1,U_2}} {\sum_{\substack{u,z_1,z_1^+\\ z_2,z_2^+}}}' \mathbf{E}[\mathbf{1}_V\mathbf{1}_T\mathbf{1}_\Phi] \\ &\qquad \times (\gamma(k_1) + \gamma(h_u - k_1 - 1) + \gamma(k_2) + \gamma(h_u - k_2 - 1) + 2) \\ &\le \frac{(4\max_{1\le k\le \delta n}\gamma(k) + 2)D^d}{c_0\sigma^4(\delta n)^{(6-d)/2}} \sum_{\substack{V_1,V_2,\\ U_1,U_2}} {\sum_{\substack{u,z_1,z_1^+\\ z_2,z_2^+}}}' \mathbf{E}[\mathbf{1}_V\mathbf{1}_T\mathbf{1}_\Phi]\,.\end{aligned} \tag{6.8}$$

We have

$$\begin{aligned}\mathbf{E}[\mathbf{1}_V\mathbf{1}_T\mathbf{1}_\Phi] = \mathbf{E}[\mathbf{1}_V\mathbf{1}_T]&\mathbf{p}^{k_1}(o, z_1)\mathbf{p}^1(z_1, z_1^+)\mathbf{p}^{h_u-k_1-1}(z_1^+, u) \\ &\times \mathbf{p}^{k_2}(x, z_2)\mathbf{p}^1(z_2, z_2^+)\mathbf{p}^{h_u-k_2-1}(z_2^+, u)\,.\end{aligned}$$

Removing the restrictions involved in the primed summation in (6.8) we can perform the convolutions of the transition probabilities and get that

$$\mathbf{E}[\mathcal{R}\mathbf{1}_{\mathcal{B}}] \le \frac{(4\max_{1\le k\le \delta n}\gamma(k) + 2)D^d}{c_0\sigma^4(\delta n)^{(6-d)/2}} \sum_{\substack{V_1,V_2,\\ U_1,U_2}} \mathbf{E}[\mathbf{1}_V\mathbf{1}_T]\mathbf{p}^{2h_u}(o, x)\,. \tag{6.9}$$

By the local central limit theorem, and due to $\|x\| \le \sqrt{\delta n}$, $h_u \ge (5/6)\delta n \ge n_1/2$, we have

$$\mathbf{p}^{2h_u}(o,x) \le \frac{C}{D^d(\delta n)^{d/2}}\,. \tag{6.10}$$

Now, fix V_1, V_2 and h_u and sum $\mathbf{1}_T$ on U_1, U_2. This number is bounded by the product of the number of vertices of $\mathcal{T}_1$ and $\mathcal{T}_2$ at height h_u, respectively. Note that this random variable is independent of $\mathbf{1}_V$, and is a product of two independent variables that have the same distribution, namely, the number of vertices of $\mathcal{T}_{\delta n, 2\delta n}$ at level h_u. The latter is stochastically smaller than the number of vertices of $\mathcal{T}_{\delta n,\infty}$ at level h_u, which has expectation $\sum_{k_1<h_u} \mathbf{E}\widetilde{\mathcal{N}}_{h_u-k_1} \le \sigma^2 \delta n$. Finally, note that

$$\sum_{V_1,V_2} \mathbf{E}[\mathbf{1}_V] = 1\,. \tag{6.11}$$

Putting together (6.9), (6.10), and (6.11) we get:

$$\mathbf{E}[\mathcal{R}\mathbf{1}_{\mathcal{B}}] \le \frac{(4\max_{1\le k\le \delta n}\gamma(k)+2)D^d}{c_0\sigma^4(\delta n)^{(6-d)/2}} \sum_{h_u=(5/6)\delta n}^{\delta n} \frac{(\sigma^2\delta n)^2}{D^d(\delta n)^{d/2}} \le C \max_{1\le k\le \delta n} \gamma(k)\,.$$

An appeal to the second part of Theorem 2.1, giving that $\mathbf{E}[\mathbf{1}_{\mathcal{B}}] \ge c_1 > 0$, concludes the proof. □

Proof of Theorem 2.2. We choose $n_5 = \max\{n_1(\mathbf{p}^1), n_9(\sigma^2, C_3, \mathbf{p}^1)\}$. Note the elementary inequality $\frac{1}{\gamma_1^{-1}+\gamma_2^{-1}} \le \frac{\gamma_1+\gamma_2}{4}$. We apply this inequality to the resistances of the two graphs "in parallel" between V_{ℓ_1} and V_{ℓ_2}: one via the backbone and one via the vertices $\mathcal{Y}_{i+1}, \dots, \mathcal{Y}_{i+K}, \mathcal{X}'_{i+K}$. The parallel law (1.3) and Lemmas 6.1 and 6.2 gives

$$\begin{aligned}
&\mathbf{E}\big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}) \,\big|\, \mathcal{G}_{\text{good}}\big] \\
&\quad \le \gamma(\ell_1 - i\delta n) + 1 + \gamma((i+K)\delta n - \ell_2 - 1) \\
&\qquad + \frac{1}{4}\left[1 + \gamma((i+1)\delta n - \ell_1 - 1) + (K-2)\gamma(\delta n) + \gamma(\ell_2 - (i+K-1)\delta n)\right] \\
&\qquad + \frac{1}{4}\big[1 + \gamma((i+1)\delta n - \ell_1 - 1) + (K-1)\gamma(\delta n) \\
&\qquad\qquad + 1 + \gamma((i+K)\delta n - \ell_2 - 1) + C_5 \max_{1\le k\le \delta n}\gamma(k)\big]\,.
\end{aligned}$$

Choosing K large with respect to C_5 concludes the proof of the theorem. □

7. Proof of Theorem 1.2

Let K_0 be the constant in Theorem 2.2. We fix $K = K_0$ for the remainder of the proof. Let

$$n_0 = \max\{n_3(\sigma^2, C_3, \mathbf{p}^1, K), n_4(\sigma^2, C_3, \mathbf{p}^1), 4k_1(\mathbf{p}^1)\}\,,$$

where n_3 and n_4 are the constants from Theorems 2.1 and 2.2 and k_1 is the constant from Proposition 1.3. Let $\delta_0 > 0$, $\alpha \in (0, 1/2)$ and $A > 0$ be constants. These will be chosen below in the order: δ_0, α, A, and among others we will require that

$$2\delta_0 \leq (K+4)^{-1}\,, \qquad 2\delta_0 \leq \min\{\delta_1, \delta_2\}\,, \tag{7.1}$$

where δ_1 is the constant from Proposition 1.3(ii) and δ_2 is the constant from Lemmas 4.1 and 4.3. Once δ_0 and α will be chosen, we choose A to satisfy:

$$A \geq n_0/\delta_0\,, \qquad A^{-1/(2\alpha)} \leq \frac{\alpha^{1/(2\alpha)}}{\sqrt{n_0}}\,. \tag{7.2}$$

We prove the theorem by induction. Since $A \geq n_0/\delta_0$, the theorem holds for all $n < n_0/\delta_0$, so we may assume $n \geq n_0/\delta_0$. Our induction hypothesis is that for all $n' < n$ and all $x \in \mathbb{Z}^d$ we have

$$\gamma(n', x) \leq A(n')^{1-\alpha}\Big(\frac{\|x\|^2}{n'} \vee 1\Big)^{\alpha}\,,$$

and given the hypothesis we prove it for n. Since $\gamma(n,x) \leq n$ it suffices to prove when $\|x\| \leq nA^{-1/(2\alpha)}$. Note that this implies

$$\|x\| \leq \frac{\alpha^{1/(2\alpha)}}{\sqrt{n_0}} n \leq \frac{n}{\sqrt{n_0}}\,.$$

Now, given such x fix

$$\delta = \min\Big\{\eta \geq \min\{\delta_0, n/\|x\|^2\} : \eta n \text{ is an integer}\Big\}\,. \tag{7.3}$$

Observe that

$$\delta \leq \delta_0 + \frac{1}{n} \leq \delta_0 + \frac{\delta_0}{n_0} \leq 2\delta_0.$$

Also note that

$$\delta n \geq \min\left\{\delta_0 n,\, \frac{n^2}{\|x\|^2}\right\} \geq n_0\,, \tag{7.4}$$

and

$$\|x\| \leq \sqrt{2n/\delta}\,, \tag{7.5}$$

so Theorem 2.1 can be applied to (x, n).

Consider the sequences

$$(0, \ldots, K), (K, \ldots, 2K), \ldots, ((N-1)K, \ldots, NK)\,,$$

$n = NK\delta n + K'\delta n + \tilde{n}$ with $0 \leq K' < K$, $\delta n \leq \tilde{n} < 2\delta n$. Fix any integer $m \geq 2n$ and define $\gamma_m(n,x)$ to be

$$\gamma_m(n,x) = \mathbf{E}_{\mathcal{T}_{n,m}}\big[R_{\mathrm{eff}}\big((o,0) \leftrightarrow \Phi(V_n)\big) \,\big|\, \Phi(V_n) = (x,n)\big]\,,$$

where we consider the resistance in the graph $\Phi(\mathcal{T}_{n,m})$, so that $\gamma(n,x) = \sup_{m \geq 2n} \gamma_m(n,x)$. We bound $\gamma_m(n,x)$ by estimating

$$\mathbf{E}\big[R_{\mathrm{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}) \,\big|\, \Phi(V_n) = (x,n)\big]\,,$$

for each $i = 0, K, 2K, \ldots, (N-1)K$ and then adding these up using the triangle inequality for resistance (1.2), also adding the estimate for the final stretch from $NK\delta n$ to n.

Fix such an i. We split the above expectation according to whether $\mathcal{A}(i) \cap \mathcal{B}(i, c_0)$ occurred. By Theorem 2.2 we have that

$$\begin{aligned}
&\mathbf{E}\big[R_{\mathrm{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\mathbf{1}_{\mathcal{A}(i)\cap\mathcal{B}(i,c_0)} \,\big|\, \Phi(V_n) = (x,n)\big] \\
&\quad \le \frac{3K \max_{1\le k\le \delta n} \gamma(k)}{4} \mathbf{P}(\mathcal{A}(i) \cap \mathcal{B}(i,c_0) \,|\, \Phi(V_n) = (x,n)) \\
&\quad \le \frac{3AK(\delta n)^{1-\alpha}}{4} \mathbf{P}(\mathcal{A}(i) \cap \mathcal{B}(i,c_0) \,|\, \Phi(V_n) = (x,n)) \,,
\end{aligned} \tag{7.6}$$

where the last inequality is due to our induction hypothesis.

We now proceed to estimate the expectation on the event that either $\mathcal{A}(i)$ or $\mathcal{B}(i, c_0)$ fail. Recall that we may write

$$\mathcal{A}(i)^c \cup \mathcal{B}(i,c_0)^c = \bigcup_{a=1}^{6} \mathcal{E}_{(\mathrm{a})} \cup (\mathcal{A}(i) \cap \mathcal{B}^c(i,c_0)) \,,$$

where $\mathcal{E}_{(\mathrm{a})}$ for $a = 1, \ldots, 6$ were defined in Section 4. To estimate the contributions coming from these terms we need the following lemmas.

Lemma 7.1. *There exists $C_6 > 0$ such that, assuming the induction hypothesis, for all $\delta n/4 \le k \le 2\delta n$ we have*

$$\bar\gamma(k; (x,n)) \le (1 + C_6\alpha)Ak^{1-\alpha} \,,$$

where $\bar\gamma$ is defined at (4.1).

Proof. By the induction hypothesis

$$\bar\gamma(k; (x,n)) \le Ak^{1-\alpha} G_1(\alpha) \,,$$

where

$$G_1(\alpha) = \sum_{y\in\mathbb{Z}^d} \frac{\mathbf{p}^k(o,y)\mathbf{p}^{n-k}(y,x)}{\mathbf{p}^n(o,x)} \Big(\frac{\|y\|^2}{k} \vee 1\Big)^{\alpha} \,.$$

We have that $G_1(0) = 1$, and that

$$G_1'(\alpha) = \sum_{y\in\mathbb{Z}^d : \|y\| > \sqrt{k}} \frac{\mathbf{p}^k(o,y)\mathbf{p}^{n-k}(y,x)}{\mathbf{p}^n(o,x)} \Big(\frac{\|y\|^2}{k}\Big)^{\alpha} \log(\|y\|^2/k) \,.$$

We bound $(\|y\|^2/k)^\alpha \log(\|y\|^2/k) \le C\|y\|^2/k$ since $\alpha \le 1/2$ and get that

$$G_1'(\alpha) \le Ck^{-1}\mathbf{E}\big[\|S(k)\|^2 \,\big|\, S(n) = x\big] \,. \tag{7.7}$$

Since $k \ge \delta n/4 \ge n_0/4 \ge k_1$ and $\|x\| \le \sqrt{2n/\delta} = \sqrt{2}n/\sqrt{\delta n} \le 4n/\sqrt{k}$, we can apply Proposition 1.3(i) to the expectation on the right hand side of (7.7). This gives that $G_1'(\alpha) \le C$, and the lemma follows. □

For the next lemma, recall the notation of Section 5.

Lemma 7.2. *There exists $C_6 > 0$ such that, assuming the induction hypothesis, for all $s' \ge s$ with $s' \ne 2, K+3$ we have*

$$\mathbf{E}\Big[R_{\mathrm{eff}}((z_{s'-1}, T_{s'-1}) \leftrightarrow (z_{s'}, T_{s'}))\mathbf{1}_{\mathcal{E}^s_{(5)}} \,\Big|\, \mathcal{G}_{\mathrm{tree}}\Big] \le (1{+}C_6\alpha)At_{s'}^{1-\alpha}\mathbf{P}(\mathcal{E}^s_{(5)} | \mathcal{G}_{\mathrm{tree}}) \,.$$

Proof. We appeal to Lemma 5.1 and use the induction hypothesis. When $s' = s$ and $s' \neq 2$, $K+3$ the required quantity is at most $At_s^{1-\alpha} G_2(\alpha)$ where

$$G_2(\alpha) = \sum_z \mathbf{q}_s(z) \sum_{y:\|y\|>\sqrt{t_s}} \frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s+T_0}(y,x-z)}{\mathbf{p}^n(o,x)} \Big(\frac{\|y\|^2}{t_{s'}} \vee 1\Big)^\alpha .$$

By (5.2) we have that

$$G_2(0) = \mathbf{P}(\mathcal{E}^s_{(5)} | \mathcal{G}_{\mathrm{tree}}) .$$

As in the previous lemma, we have

$$G_2'(\alpha) \leq \sum_z \mathbf{q}_s(z) \sum_{y:\|y\|>\sqrt{t_s}} \frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s+T_0}(y,x-z)}{\mathbf{p}^n(o,x)} \frac{\|y\|^2}{t_s} . \tag{7.8}$$

We multiply and divide by $\mathbf{p}^{n-T_{s-1}+T_0}(o,z-x)$, and rewrite the expression in the right hand side of (7.8) as

$$\begin{aligned} &\frac{1}{t_s} \sum_z \mathbf{q}_s(z) \frac{\mathbf{p}^{n-T_{s-1}+T_0}(o,z-x)}{\mathbf{p}^n(o,x)} \\ &\qquad \times \sum_{y:\|y\|>\sqrt{t_s}} \|y\|^2 \frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s+T_0}(y,x-z)}{\mathbf{p}^{n-T_{s-1}+T_0}(o,x-z)} . \end{aligned} \tag{7.9}$$

We now fix z and want to apply Proposition 1.3(ii) to the sum over y in (7.9). Observe that

$$\begin{aligned} \|x-z\| \leq \|x\| + \|z\| \leq \sqrt{2n/\delta} + (K+4)\sqrt{\delta n} &= n\left(\frac{\sqrt{2}}{\sqrt{\delta n}} + \frac{\delta(K+4)}{\sqrt{\delta n}}\right) \\ \leq n\frac{3}{\sqrt{\delta n}} , \end{aligned}$$

where in the last step we used (7.1). This implies that $\|x-z\| \leq 3n/\sqrt{\delta n} \leq 3n/\sqrt{t_s}$. We also have $t_s \geq \delta n/4 \geq n_0/4 \geq k_1$, where k_1 is the constant chosen in Proposition 1.3, and $t_s \leq \delta n \leq \delta_1 n$, due to (7.1). Hence we can apply Proposition 1.3(ii). We get that

$$\begin{aligned} &\sum_{y:\|y\|>\sqrt{t_s}} \|y\|^2 \frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s+T_0}(y,x-z)}{\mathbf{p}^{n-T_{s-1}+T_0}(o,x-z)} \\ &\quad = \mathbf{E}\Big[\|S(t_s)\|^2 \mathbf{1}_{\|S(t_s)\|>\sqrt{t_s}} \,\Big|\, S(n-T_{s-1}+T_0) = x-z\Big] \\ &\quad \leq Ct_s \mathbf{P}\Big(\|S(t_s)\| > \sqrt{t_s} \,\Big|\, S(n-T_s+T_0) = x-z\Big) \\ &\quad = Ct_s \sum_{y:\|y\|>\sqrt{t_s}} \frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s+T_0}(y,x-z)}{\mathbf{p}^{n-T_{s-1}+T_0}(o,x-z)} . \end{aligned}$$

Substituting this back into (7.9) and cancelling the factors $\mathbf{p}^{n-T_{s-1}+T_0}(o, z-x)$, we get

$$
\begin{aligned}
G_2'(\alpha) &\le C \sum_{\substack{z,y:\\ \|y\|>\sqrt{t_s}}} \mathbf{q}_s(z) \frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s+T_0}(y, x-z)}{\mathbf{p}^n(o,x)} \\
&= C\mathbf{P}(\mathcal{E}^s_{(5)}|\mathcal{G}_{\mathrm{tree}}) \,.
\end{aligned}
$$

This gives the statement of the lemma in the case $s' = s$.

The case $s' > s$ is similar. We appeal to the last statement of Lemma 5.1, and obtain that the required quantity is at most $At_{s'}^{1-\alpha}G_3(\alpha)$ where

$$
G_3(\alpha) = \sum_{\substack{z,y_s,y:\\ \|y_s\|>\sqrt{t_s}}} \mathbf{q}_s(z) \frac{\mathbf{p}^{t_s}(o,y_s)\mathbf{p}^{t_{s'}}(o,y)\mathbf{p}^{n-t_{s'}-T_s}(y, x-z-y_s)}{\mathbf{p}^n(o,x)} \Big(\frac{\|y\|^2}{t_{s'}} \vee 1\Big)^{\alpha} .
$$

By (5.2) we see (performing the sum over y) that $G_3(0) = \mathbf{P}(\mathcal{E}^s_{(5)}|\mathcal{G}_{\mathrm{tree}})$. The derivative $G_3'(\alpha)$ can be analyzed similarly to G_2', this time using Proposition 1.3(iii). This yields $G_3'(\alpha) \le C\mathbf{P}(\mathcal{E}^s_{(5)}|\mathcal{G}_{\mathrm{tree}})$, and proves the statement of the lemma in the case $s' > s$. $\square$

We now proceed with bounding the resistance given $\mathcal{A}(i)^c \cup \mathcal{B}(i, c_0)^c$. Lemmas 4.1 and 7.1 and the induction hypothesis give that

$$
\begin{aligned}
&\mathbf{E}\Big[R_{\mathrm{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \,\Big|\, \mathcal{E}_{(1)}, \Phi(V_n) = (x,n)\Big] \\
&\qquad \le A(K + C_4\delta)(1 + C_6\alpha)(\delta n)^{1-\alpha} \,.
\end{aligned}
$$

Lemmas 4.2 and 7.1 give

$$
\begin{aligned}
&\mathbf{E}\Big[R_{\mathrm{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \,\Big|\, \mathcal{E}_{(2)},\, \ell_1,\, \Phi(V_n) = (x,n)\Big] \\
&\qquad \le A(1 + C_6\alpha)\Big[(\ell_1 - i\delta n)^{1-\alpha} + ((i+1)\delta n - \ell_1)^{1-\alpha} \\
&\qquad\qquad + (K-1)(\delta n)^{1-\alpha}\Big] + 1 \,.
\end{aligned}
$$

Since $\frac{\ell_1}{\delta n} - i \in [1/4, 3/4]$,

$$
(\ell_1 - i\delta n)^{1-\alpha} + ((i+1)\delta n - \ell_1)^{1-\alpha} \le (1 + C_7\alpha)(\delta n)^{1-\alpha} ,
$$

where $C_7 = (1/2)\log 4 + \sqrt{(3/4)}\log(4/3)$. Hence

$$
\begin{aligned}
&\mathbf{E}\Big[R_{\mathrm{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \,\Big|\, \mathcal{E}_{(2)},\, \ell_1,\, \Phi(V_n) = (x,n)\Big] \\
&\qquad \le A(1 + (C_6 + C_7 + C_6C_7)\alpha)K(\delta n)^{1-\alpha} + 1 \,.
\end{aligned}
$$

Lemmas 4.3 and 7.1 give

$$\begin{aligned}\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{E}_{(3)},\, \ell_1,\, \Phi(V_n)=(x,n)\Big] \\ \le A(1+C_6\alpha)\Big[(\ell_1 - i\delta n)^{1-\alpha} + ((i+1)\delta n - \ell_1)^{1-\alpha} \\ + (K-1+C_4\delta)(\delta n)^{1-\alpha}\Big] + 1 \\ \le A(K+C_4\delta+C_7\alpha)(1+C_6\alpha)(\delta n)^{1-\alpha} + 1\,.\end{aligned}$$

Lemmas 4.4 and 7.1 give

$$\begin{aligned}\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{E}_{(4)},\, \ell_1,\, \ell_2,\, \Phi(V_n)=(x,n)\Big] \\ \le A(1+C_6\alpha))\Big[(\ell_1 - i\delta n)^{1-\alpha} + ((i+1)\delta n - \ell_1)^{1-\alpha} + (K-2)(\delta n)^{1-\alpha} \\ + (\ell_2 - (i+K-1)\delta n)^{1-\alpha} + ((i+K)\delta n - \ell_2)^{1-\alpha}\Big] + 2 \\ \le A(1+C_6\alpha))(K+2C_7\alpha)(\delta n)^{1-\alpha} + 2\,.\end{aligned}$$

Lemmas 5.1, 7.2 and the induction hypothesis give that for any $s = 1, \ldots, K+4$ and $s' = 1, \ldots, K+4$ we have that

$$\begin{aligned}\mathbf{E}\Big[R_{\text{eff}}((z_{s'-1}, T_{s'-1}) \leftrightarrow (z_{s'}, T_{s'}))\,\Big|\,\mathcal{E}^s_{(5)}, \mathcal{G}_{\text{tree}}\Big] \\ \le \begin{cases} 1 & \text{if } s' = 2, K+3, \\ A(t_{s'})^{1-\alpha} & \text{if } s' < s, s' \neq 2, K+3, \\ A(1+C_6\alpha)(t_{s'})^{1-\alpha} & \text{if } s' \ge s, s' \neq 2, K+3. \end{cases}\end{aligned}$$

By the triangle inequality for resistance we get that for all $s = 1, \ldots, K+4$

$$\begin{aligned}\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{E}^s_{(5)}, \mathcal{G}_{\text{tree}}\Big] \\ \le AK(1+C_6\alpha)(1+2C_7\alpha)(\delta n)^{1-\alpha} + 2\,.\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{E}_{(5)},\, \ell_1,\, \ell_2,\, \Phi(V_n)=(x,n)\Big] \\ \le AK(1+C_6\alpha)(1+2C_7\alpha)(\delta n)^{1-\alpha} + 2\,.\end{aligned}$$

By Lemma 5.2 and the induction hypothesis (recall $\mathcal{E}_{(7)} = \mathcal{A}(i) \cap \mathcal{B}(i, c_0)^c$):

$$\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{E}_{(6)} \cup \mathcal{E}_{(7)}, \mathcal{G}_{\text{tree}}\Big] \le A(K+2C_7\alpha)(\delta n)^{1-\alpha} + 2\,.$$

Putting these together gives that there exists $C_8 = C_8(K) > 0$ such that

$$\begin{aligned}\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{A}(i)^c \cup \mathcal{B}(i,c_0)^c,\, \Phi(V_n)=(x,n)\Big] \\ \le A(K+C_8(\delta+\alpha))(\delta n)^{1-\alpha} + 2\,.\end{aligned}$$

This together with (7.6) yields

$$\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\Phi(V_n) = (x,n)\Big] \\
&\quad \le \frac{3AK(\delta n)^{1-\alpha}}{4}\mathbf{P}(\mathcal{A}(i) \cap \mathcal{B}(i,c_0) \mid \Phi(V_n) = (x,n)) \\
&\quad + \Big[A(K + C_8(\delta+\alpha))(\delta n)^{1-\alpha} + 2\Big]\mathbf{P}(\mathcal{A}(i)^c \cup \mathcal{B}(i,c_0)^c \mid \Phi(V_n) = (x,n)) \,.
\end{aligned}$$

By Theorem 2.1 there exists a constant $c = c(K) \in (0,1)$ such that the last quantity is at most

$$A(\delta n)^{1-\alpha}\Big[\frac{c3K}{4} + (1-c)(K + C_8(\delta+\alpha) + 2\alpha)\Big] \,,$$

where we bounded the pesky remaining term $2A^{-1}(\delta n)^{\alpha-1}$ by

$$2A^{-1}(\delta n)^{\alpha-1} \le 2\alpha n_0^{-\alpha} n_0^{\alpha-1} \le 2\alpha \,.$$

using the second inequality in (7.2) and (7.4). We now choose δ_0 and α (depending only on $K = K_0$). In addition to the already required (7.1), let δ_0 satisfy:

$$2\delta_0 \le \frac{c}{16(C_8+2)}\,, \qquad K(2\delta_0)(1+C_6) + 4\delta_0(1+C_6) < \frac{c}{16}\,, \tag{7.10}$$

Let $\alpha > 0$ satisfy:

$$\alpha \le \delta_0\,, \qquad \Big(1 - \frac{c}{16}\Big)\delta_0^{-\alpha} \le 1\,. \tag{7.11}$$

The first condition on δ_0 in (7.10) gives that

$$\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\Phi(V_n) = (x,n)\Big] \\
&\quad \le AK(1-c/8)(\delta n)^{1-\alpha} \\
&\quad \le An^{1-\alpha}\delta^{-\alpha}K\delta(1-c/8)\,,
\end{aligned} \tag{7.12}$$

for $i = 0, K, \ldots, (N-1)K$. For the final stretch, Lemmas 5.3 and 7.1 and the induction hypothesis gives:

$$\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_{j^{\text{last}}}) \leftrightarrow \Phi(V_n))\,\Big|\,\Phi(V_n) = (x,n)\Big] \\
&\quad \le K'A(\delta n)^{1-\alpha}(1+C_6\alpha) + A(\tilde{n})^{1-\alpha}(1+C_6\alpha)\,.
\end{aligned} \tag{7.13}$$

Using $K' \le K$ and $\tilde{n} \le 2\delta n$ and the second requirement on δ_0 in (7.10), the right hand side of (7.13) is at most

$$\begin{aligned}
&An^{1-\alpha}\delta^{-\alpha}\big(K\delta(1+C_6) + 2\delta(1+C_6)\big) \\
&\quad \le An^{1-\alpha}\delta^{-\alpha}(c/16)\,.
\end{aligned}$$

We sum (7.12) over all sequences using the triangle inequality and add (7.13). This gives

$$\begin{aligned}\gamma(n,x) &= \sup_{m\geq 2n} \gamma_m(n,x) \\ &\leq An^{1-\alpha}\delta^{-\alpha}\Big[NK\delta(1-c/8)+c/16\Big] \\ &\leq An^{1-\alpha}\delta^{-\alpha}(1-c/16)\,,\end{aligned} \tag{7.14}$$

where we used $NK\delta \leq 1$.

To conclude, note that if in the definition (7.3) of δ we have $\delta_0 \leq n/\|x\|^2$, then $\delta^{-\alpha} \leq \delta_0^{-\alpha}$, and due to the choice of α, (7.14) yields $\gamma(n,x) \leq An^{1-\alpha}$. If we have $n/\|x\|^2 < \delta_0$, then $\delta^{-\alpha} \leq (\|x\|^2/n)^\alpha$, and (7.14) yields $\gamma(n,x) \leq An^{1-\alpha}(\|x\|^2/n)^\alpha$ This concludes the induction and our proof. □

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